2 CLASSICAL DISPERSION THEORY

2.1 INTRODUCTION

An understanding of lasers requires some knowledge of the way in which light and matter interact. Many features of this interaction may be explained using a simple classical model. In this chapter and the next we describe this model and use it to introduce some concepts that are especially important for lasers.

We assume that the reader has previously encountered Maxwell's equations, at least briefly, and understands that they provide the most fundamental description of electric and magnetic fields. For a neutral dielectric medium (one with no free charges) Maxwell's equations are

$$\nabla \cdot \mathbf{D} = 0 \tag{2.1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1.3}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{2.1.4}$$

We will be interested only in nonmagnetic media, for which

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{2.1.5}$$

where $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$. The electric displacement **D** is defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{2.1.6}$$

where $1/4\pi\epsilon_0 = 8.9874 \times 10^9\ \mathrm{N\text{-}m^2/C^2}$ and the polarization **P** is the electric dipole moment per unit volume of the medium. **P** is the only term in the Maxwell equations relating directly to the medium.

Applying the curl operation to both sides of Eq. (2.1.3), we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{2.1.7}$$

Now we use the general identity (see Problem 2.1)

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{2.1.8}$$

of vector calculus, together with (2.1.5) and the Maxwell equation (2.1.4), to write

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} \tag{2.1.9}$$

Finally we use the definition (2.1.6) of **D** and rearrange terms:

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{2.1.10}$$

Here we have used the fact that

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \tag{2.1.11}$$

where $c = 2.998 \times 10^8$ m/sec is the velocity of light in vacuum.[1]

Equation (2.1.10) is a partial differential equation with independent variables x, y, z, and t. It tells us how the electric field depends on the electric dipole moment density **P** of the medium. We will be particularly interested in *transverse* fields (sometimes called *solenoidal* or *radiation* fields). Such fields satisfy

$$\nabla \cdot \mathbf{E} = 0 \tag{2.1.12}$$

Transverse fields therefore satisfy the inhomogeneous wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{2.1.13}$$

This is the fundamental electromagnetic field equation for our purposes. In order to make any use of it we must somehow specify the polarization **P**. This cannot be done solely within the framework of the Maxwell equations, for **P** is a property of the material medium that the field **E** propagates in. We need to know how a dipole moment density **P** is produced in the medium. For this purpose we will introduce in the next section a theory of dielectric media.

First, however, we will finish this section with a discussion of solutions to the homogeneous (free-space) wave equation, which applies when there is no polarization present. Laser resonator theory is based on the free-space wave equation and free-space solutions. Such solutions are useful, even though lasers do not operate in vacuum, because most laser media are optically homogeneous.

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[1] For convenience we include inside the cover of the book a table of physical constants that appear frequently in our study of lasers.

We will consider only the case of a rectangular cavity, as sketched in Figure 1.10. We also imagine we have perfectly reflecting walls; then the components of the electric field parallel to the walls must vanish on the walls. The electric field inside the cavity satisfies the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0 \tag{2.1.14}$$

For a monochromatic field of angular frequency $\omega = 2\pi\nu$, we may use the complex-field representation (where the physical electric field is understood to be the real part of the right-hand side):

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})\, e^{-i\omega t} \tag{2.1.15}$$

and (2.1.14) becomes

$$\nabla^2 \mathbf{E}_0(\mathbf{r}) + k^2 \mathbf{E}_0(\mathbf{r}) = 0, \qquad k \equiv \omega/c \tag{2.1.16}$$

That is,

$$(\nabla^2 + k^2)\, E_{0x}(\mathbf{r}) = 0 \tag{2.1.17}$$

and likewise for the y and z components.

To solve (2.1.17), it is convenient to use the method of separation of variables. Separation means that the solution is written in a factored form:

$$E_{0x}(x, y, z) = F(x)\, G(y)\, H(z) \tag{2.1.18}$$

and then inserted in (2.1.17). After carrying out the differentiations required by $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, we divide through by the product FGH and obtain

$$\frac{1}{F}\frac{d^2F}{dx^2} + \frac{1}{G}\frac{d^2G}{dy^2} + \frac{1}{H}\frac{d^2H}{dz^2} + k^2 = 0 \tag{2.1.19}$$

Since each of the first three terms on the left side is a function of a different independent variable, Eq. (2.1.19) [and hence (2.1.17)] can only be true for all x, y, and z if each term is separately constant, i.e.,

$$\frac{1}{F}\frac{d^2F}{dx^2} = -k_x^2 \tag{2.1.20a}$$

$$\frac{1}{G}\frac{d^2G}{dy^2} = -k_y^2 \tag{2.1.20b}$$

$$\frac{1}{H}\frac{d^2H}{dz^2} = -k_z^2 \tag{2.1.20c}$$

with

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{2.1.21}$$

The boundary condition that the tangential component of the electric field vanishes on the cavity walls means that

$$E_{0x}(x, y = 0, z) = E_{0x}(x, y = L_y, z) = 0 \tag{2.1.22a}$$

and

$$E_{0x}(x, y, z = 0) = E_{0x}(x, y, z = L_z) = 0 \tag{2.1.22b}$$

or

$$G(0) = G(L_y) = 0 \tag{2.1.23a}$$

$$H(0) = H(L_z) = 0 \tag{2.1.23b}$$

A solution of (2.1.20b) satisfying the boundary condition $G(0) = 0$ is

$$G(y) = \sin k_y y \tag{2.1.24}$$

In order to satisfy $G(L_y) = 0$ as well, we must have $\sin k_y L_y = 0$, or in other words

$$k_y L_y = m\pi, \qquad m = 0, 1, 2, \ldots \tag{2.1.25a}$$

In exactly the same way we find that solutions of Eq. (2.1.20c) satisfying (2.1.23b) are only possible if

$$k_z L_z = n\pi, \qquad n = 0, 1, 2, \ldots \tag{2.1.25b}$$

Finally, consideration of the equations for the y and z components of $\mathbf{E}_0(\mathbf{r})$, together with the appropriate boundary conditions, shows that allowed solutions for $\mathbf{E}_0(\mathbf{r})$ must satisfy (2.1.25a), (2.1.25b), and

$$k_x L_x = l\pi, \qquad l = 0, 1, 2, \ldots \tag{2.1.25c}$$

The full solutions for the components of $\mathbf{E}(\mathbf{r}, t)$ satisfying Maxwell's equations and the boundary conditions inside the cavity are (Problem 2.2)

$$E_x(x, y, z, t) = A_x e^{-i\omega t} \cos\frac{l\pi x}{L_x} \sin\frac{m\pi y}{L_y} \sin\frac{n\pi z}{L_z} \tag{2.1.26a}$$

$$E_y(x, y, z, t) = A_y e^{-i\omega t} \sin\frac{l\pi x}{L_x} \cos\frac{m\pi y}{L_y} \sin\frac{n\pi z}{L_z} \tag{2.1.26b}$$

$$E_z(x, y, z, t) = A_z e^{-i\omega t} \sin\frac{l\pi x}{L_x} \sin\frac{m\pi y}{L_y} \cos\frac{n\pi z}{L_z} \tag{2.1.26c}$$

where the coefficients A_x, A_y, and A_z must satisfy the condition (Problem 2.3)

$$\frac{l}{L_x}A_x + \frac{m}{L_y}A_y + \frac{n}{L_z}A_z = 0 \tag{2.1.27}$$

implied by the Maxwell equation $\nabla \cdot \mathbf{E} = 0$, valid in the empty cavity.

From Eqs. (2.1.21) and (2.1.25) we have

$$k^2 = \pi^2\left(\frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2}\right) \tag{2.1.28}$$

The possible modes of the rectangular closed cavity have allowed frequencies determined by (2.1.28) and $k = \omega/c = 2\pi\nu/c$ [recall (2.1.16)]:

$$\nu = \nu_{lmn} = \frac{c}{2}\left(\frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2}\right)^{1/2} \tag{2.1.29}$$

This formula is a generalization of (1.3.7), in which case the cavity was assumed cubical ($L_x = L_y = L_z = L$). The same mode-density formulas given in Chapter 1 are valid here, independent of cavity shape, as long as L_x, L_y, L_z are all much larger than $\lambda = c/\nu$. That is, the number of modes in the frequency interval $[\nu, \nu + d\nu]$ is

$$dN_\nu = \frac{8\pi\nu^2}{c^3} V\, d\nu \tag{2.1.30a}$$

and in the wavelength range $d\lambda$ the number is (Problem 2.4)

$$dN_\lambda = 8\pi\left(\frac{V}{\lambda^3}\right)\frac{d\lambda}{\lambda} \tag{2.1.30b}$$

• It is not difficult to derive the important mode-density formulas for dN_ν and dN_λ given above. First of all, the number of modes available in a cavity is infinite. This is clear because in (2.1.29), for example, an infinite number of values are permitted for any of the three mode indices l, m, n. However, the number of modes *whose frequency lies in the neighborhood* $d\nu$ *of a given value* ν is finite. This number is related to the number of modes *whose frequency is less than* ν, and it is this number we will determine first.

The number of modes we want is the number of terms in the triple sum:

$$N = \sum_l \sum_m \sum_n \tag{2.1.31}$$

where the upper limits on the sums are determined by the maximum frequency to be included. The simplest approach to this problem is to stipulate that the cavity is much larger than a typical wavelength (obviously true for realistic cavities and optical wavelengths). Then the discrete nature of the sum is not important and we can rewrite the sum as a triple integral:

$$N = \int dl \int dm \int dn \tag{2.1.32}$$

In addition, for a large cavity the shape is not very important in determining the *number* of modes (although critical for the spatial characteristics of the modes, of course), so for our present purpose we can just as well assume the simplest shape—a cube with sides equal to L. For a cubical cavity (2.1.29) becomes

$$\left(\frac{2L}{c}\right)^2 \nu^2 = l^2 + m^2 + n^2 \tag{2.1.33}$$

It is a useful trick to regard the triplet (l, m, n) as the components of a fictitious vector $\mathbf{q}$:

$$\mathbf{q} = \mathbf{i}l + \mathbf{j}m + \mathbf{k}n \tag{2.1.34a}$$

with magnitude

$$\mathbf{q}^2 = q^2 = l^2 + m^2 + n^2 \tag{2.1.34b}$$

Then the triple integral can be denoted

$$N = \iiint d^3q \tag{2.1.35}$$

Equation (2.1.33) indicates that ν depends only on the length, but not the orientation, of the vector $\mathbf{q}$. Thus we rewrite the mode integral in spherical coordinates:

$$N = \iiint q^2\, dq \sin\theta_q\, d\theta_q\, d\phi_q \tag{2.1.36}$$

and carry out the integrations to get:

$$\begin{aligned} N &= \frac{4\pi}{8} \int q^2\, dq \\ &= \frac{4\pi}{8} \frac{q^3}{3} \end{aligned} \tag{2.1.37}$$

Here the factor 4π is the result of the angular integration (4π total solid angle) and the $\frac{1}{8}$ is due to the restriction on the original integers l, m, n to be positive, so that only the vectors $\mathbf{q}$ in the positive octant of the integration (2.1.35) should be counted as corresponding to physical modes.

In (2.1.37) q is the length of the vector $\mathbf{q}$ compatible with the given frequency ν. From (2.1.33) it is clear that $q = (2L/c)\,\nu$, so we finally get

$$\begin{aligned} N_\nu &= \frac{\pi}{6} \left(\frac{2L}{c}\right)^3 \nu^3 \\ &= 4\pi \frac{\nu^3}{3c^3} V \end{aligned} \tag{2.1.38}$$

where $V = L^3$ is the cavity volume.

Since our derivation of (2.1.38) did not take account of the polarization of the cavity modes, we are still free to choose independently any *two* polarizations (see Problem 2.3). Thus we have

$$N_\nu = \frac{8\pi\nu^3}{3c^3} V \qquad \text{(all polarizations)} \tag{2.1.39}$$

for the number of possible cavity modes of frequency less than ν, counting all polarizations.

The number of possible field modes in the frequency interval from ν to $\nu + d\nu$ is therefore

$$dN_\nu = \frac{8\pi\nu^2}{c^3} V\, d\nu \tag{2.1.40}$$

which is exactly the formula used in Section 1.3 and reproduced in (2.1.30a). The formula for the density dN_λ given in (2.1.30b) is obtained from dN_ν by application of the relation $\nu = c/\lambda$ (see Problem 2.4). •

2.2 THE ELECTRON OSCILLATOR MODEL

In classical physics the motion of a particle is described by Newton's second law. For a charged particle in an electromagnetic field the force referred to in Newton's second law is the Lorentz force

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.2.1}$$

where e and $\mathbf{v}$ are the charge and velocity, respectively, of the particle.

We are interested in the atoms that play roles in laser action. These are atoms in dielectrics such as crystals, vapors, glasses, and liquids. Atoms are neutral objects, but (2.2.1) applies to the individual protons and electrons in these atoms. The interaction of atomic protons and electrons with light can be treated very accurately in most cases with classical laws and concepts. The quantum-mechanical basis for our classical treatment, and quantum corrections to several classical formulas, are discussed in Chapter 6.

The electron has mass m_e and charge e (a negative number), and the oppositely charged core of the atom ("nucleus") has mass m_n and charge $-e$. The nucleus exerts a binding force $\mathbf{F}_{en}$ on the electron, depending on the relative separation $\mathbf{r}_{en} = \mathbf{r}_e - \mathbf{r}_n$, as shown in Figure 2.1. The electron also exerts a force $\mathbf{F}_{ne}$ on the nucleus, and according to Newton's third law,

$$\mathbf{F}_{ne}(\mathbf{r}_{en}) = -\mathbf{F}_{en}(\mathbf{r}_{en}) \tag{2.2.2}$$

The Newton equations of motion for the electron and nucleus are therefore

$$m_e \frac{d^2\mathbf{r}_e}{dt^2} = e\mathbf{E}(\mathbf{r}_e, t) + \mathbf{F}_{en}(\mathbf{r}_{en}) \tag{2.2.3a}$$

and

$$m_n \frac{d^2\mathbf{r}_n}{dt^2} = -e\mathbf{E}(\mathbf{r}_n, t) + \mathbf{F}_{ne}(\mathbf{r}_{en}) \tag{2.2.3b}$$

In writing Eqs. (2.2.3) we have dropped the magnetic contribution to the Lorentz force. Because optical phenomena do not normally involve relativistic particle velocities, we can safely disregard the magnetic force hereafter.

The interaction of electromagnetic fields with charges is mainly determined by the acceleration of the charges. The nucleus is so massive compared to an electron that its acceleration is generally negligible. In this case only the electron equation is needed. The binding force $\mathbf{F}_{en}$ is strong enough to restrict the atomic electrons to small excursions about the (approximately stationary) nucleus. Thus we can write $\mathbf{r}_e = \mathbf{r}_n + \mathbf{x}$, where $\mathbf{x}$ is an atomic dimension ($|\mathbf{x}| \lesssim 10$ Å $= 1$ nm $= 10^{-9}$ m, 1 nanometer). The electric radiation field varies on the scale of an optical wavelength ($\lambda \approx 6000$ Å $= 600$ nm for yellow light) and is not sensitive to variations as small as $|\mathbf{x}|$, so we have $\mathbf{E}(\mathbf{r}_e, t) \approx \mathbf{E}(\mathbf{r}_n, t)$.

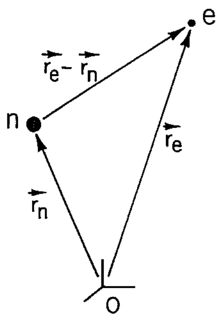

Figure 2.1 The position vectors $\mathbf{r}_e$ and $\mathbf{r}_n$ of the electron and nucleus, measured from some origin O. By Newton's third law the force $\mathbf{F}_{en}(\mathbf{r}_e - \mathbf{r}_n)$ exerted by the nucleus on the electron is equal in magnitude but opposite in direction to the force $\mathbf{F}_{ne}(\mathbf{r}_e - \mathbf{r}_n)$ exerted by the electron on the nucleus.

Within these approximations it is easy to see that the effective interaction equation replacing Eqs. (2.2.3) is

$$m \frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E}(\mathbf{R}, t) + \mathbf{F}_{en}(\mathbf{x}) \tag{2.2.4}$$

Here we have dropped the subscript e from the electron mass and have written $\mathbf{R}$ for the position of the stationary nucleus. Actually, $\mathbf{x}$ and $\mathbf{R}$ are the relative coordinate and center-of-mass coordinate of the electron–nucleus pair, and m is the associated reduced mass. These terms are defined in the black-dot section below. For our purposes it is accurate to continue to think of $\mathbf{R}$ as the position of the nucleus and m as the electron mass. Only in exceptional cases in which the two charges have nearly equal mass, such as the positronium atom (an atom in which the nucleus is a positron, i.e., an anti-electron, rather than a proton), would significant corrections be required.

Note that the electric force in (2.2.4) can be written in terms of a potential $V(\mathbf{x}, \mathbf{R}, t)$. If we define

$$V(\mathbf{x}, \mathbf{R}, t) = -e\mathbf{x} \cdot \mathbf{E}(\mathbf{R}, t) \tag{2.2.5}$$

then we have

$$\begin{aligned} e\mathbf{E}(\mathbf{R}, t) &= -\nabla_{\mathbf{x}} \left[-e\mathbf{x} \cdot \mathbf{E}(\mathbf{R}, t) \right] \\ &= -\nabla_{\mathbf{x}} V(\mathbf{x}, \mathbf{R}, t) \end{aligned} \tag{2.2.6}$$

where $\nabla_{\mathbf{x}}$ indicates the gradient with respect to the coordinate $\mathbf{x}$. The potential V is proportional to $e\mathbf{x} = \mathbf{p} = e(\mathbf{r}_e - \mathbf{r}_n)$, which is the *electric dipole moment* of the atom. Thus the equation of motion (2.2.4) is frequently identified as belonging to the *dipole approximation*. Recall that the main approximation is to take $\mathbf{E}$ to depend only on the nuclear position $\mathbf{R}$. This was justified by the tightness of the atomic binding, which restricts $\mathbf{x}$ to a range much smaller than the wavelength of the radiation described by $\mathbf{E}$.

• The center of mass of the electron–nucleus system is defined to be

$$\mathbf{R} = \frac{m_e\mathbf{r}_e + m_n\mathbf{r}_n}{M} \tag{2.2.7a}$$

where $M = m_e + m_n$ and $\mathbf{x}$ is the electron coordinate relative to the nucleus,

$$\mathbf{x} = \mathbf{r}_{en} \tag{2.2.7b}$$

in terms of which

$$\mathbf{r}_e = \mathbf{R} + \frac{m_n}{M}\mathbf{x} \tag{2.2.8a}$$

$$\mathbf{r}_n = \mathbf{R} - \frac{m_e}{M}\mathbf{x} \tag{2.2.8b}$$

Then Eqs. (2.2.3) may be written as

$$m_e\frac{d^2\mathbf{R}}{dt^2} + m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E}\left(\mathbf{R} + \frac{m_n}{M}\mathbf{x}, t\right) + \mathbf{F}_{en}(\mathbf{x}) \tag{2.2.9a}$$

$$m_n\frac{d^2\mathbf{R}}{dt^2} - m\frac{d^2\mathbf{x}}{dt^2} = -e\mathbf{E}\left(\mathbf{R} - \frac{m_e}{M}\mathbf{x}, t\right) + \mathbf{F}_{ne}(\mathbf{x}) \tag{2.2.9b}$$

where

$$m = \frac{m_e m_n}{M} = \frac{m_e m_n}{m_e + m_n} \tag{2.2.10}$$

is called the *reduced mass* of the electron–nucleus system.

By adding and subtracting Eqs. (2.2.9), and using (2.2.2), we obtain the equations of motion

$$M\frac{d^2\mathbf{R}}{dt^2} = e\left[\mathbf{E}\left(\mathbf{R} + \frac{m_n}{M}\mathbf{x}, t\right) - \mathbf{E}\left(\mathbf{R} - \frac{m_e}{M}\mathbf{x}, t\right)\right] \tag{2.2.11a}$$

and

$$m\frac{d^2\mathbf{x}}{dt^2} = \frac{e}{2}\left[\mathbf{E}\left(\mathbf{R} + \frac{m_n}{M}\mathbf{x}, t\right) + \mathbf{E}\left(\mathbf{R} - \frac{m_e}{M}\mathbf{x}, t\right)\right] + \mathbf{F}_{en}(\mathbf{x}) + \tfrac{1}{2}(m_n - m_e)\frac{d^2\mathbf{R}}{dt^2} \tag{2.2.11b}$$

Equation (2.2.11a) describes the motion of the center of mass of the atom. In the absence of an external field, the center of mass moves with constant velocity. Equation (2.2.11b) describes the motion of the relative coordinate $\mathbf{x}$ of the electron–nucleus system.

We have already mentioned that optical radiation is characterized by wavelengths that are a few thousand Ångstrom units ($1\ \text{Å} = 10^{-10}$ m) or larger, and the electron–nucleus separations in atoms are typically only 1–10 Å in size. The extreme disparity of these sizes is the basis of a fundamental approximation called the dipole approximation. The dipole approximation arises from the leading terms of a Taylor series expansion of the type

$$F(X + \delta X) = F(X) + \delta X F'(X) + \tfrac{1}{2}(\delta X)^2 F''(X) + \cdots$$

applied to the electric field vectors in (2.2.11). The vector analog of the Taylor series gives

$$\mathbf{E}\left(\mathbf{R} - \frac{m_e}{M}\mathbf{x}, t\right) = \mathbf{E}(\mathbf{R}, t) - \frac{m_e}{M}\mathbf{x}\cdot\nabla\mathbf{E}(\mathbf{R}, t) + \cdots \tag{2.2.12a}$$

and

$$\mathbf{E}\left(\mathbf{R} + \frac{m_n}{M}\mathbf{x}, t\right) = \mathbf{E}(\mathbf{R}, t) + \frac{m_n}{M}\mathbf{x}\cdot\nabla\mathbf{E}(\mathbf{R}, t) + \cdots \tag{2.2.12b}$$

and if we retain only the first two terms in the Taylor series (2.2.12), then Eqs. (2.2.11) become

$$M\frac{d^2\mathbf{R}}{dt^2} \approx e\mathbf{x}\cdot\nabla\mathbf{E}(\mathbf{R}, t) \tag{2.2.13a}$$

$$m\frac{d^2\mathbf{x}}{dt^2} \approx e\mathbf{E}(\mathbf{R}, t) + \left(\frac{m_n - m_e}{M}\right)e\mathbf{x}\cdot\nabla\mathbf{E}(\mathbf{R}, t) + \mathbf{F}_{en}(\mathbf{x}) \tag{2.2.13b}$$

We have already mentioned that the vector

$$\mathbf{p} = e\mathbf{x} \tag{2.2.14}$$

is the electric dipole moment of the electron–nucleus pair. In terms of $\mathbf{p}$ Eq. (2.2.13a) is

$$M\frac{d^2\mathbf{R}}{dt^2} = \mathbf{p}\cdot\nabla\mathbf{E}(\mathbf{R}, t) = \mathbf{F} = -\nabla_{\mathbf{R}}V(\mathbf{x}, \mathbf{R}, t) \tag{2.2.15}$$

where $\nabla_{\mathbf{R}}$ denotes the gradient with respect to the coordinate $\mathbf{R}$ and

$$V(\mathbf{x}, \mathbf{R}, t) = -\mathbf{p}\cdot\mathbf{E}(\mathbf{R}, t) \tag{2.2.16}$$

is the potential energy of an electric dipole $\mathbf{p}$ at the point $\mathbf{R}$ in an electric field. It was already identified in (2.2.5) above. Finally, we retain only the leading $\mathbf{E}$ term on the right-hand side of (2.2.13b) and obtain

$$m\frac{d^2\mathbf{x}}{dt^2} \approx e\mathbf{E}(\mathbf{R}, t) + \mathbf{F}_{en}(\mathbf{x}) = -\nabla_{\mathbf{x}}V + \mathbf{F}_{en}(\mathbf{x}) \qquad \text{(electric-dipole approximation)} \tag{2.2.17}$$

which is Eq. (2.2.4) again, this time with m, $\mathbf{x}$ and $\mathbf{R}$ carefully defined. •

For most of our purposes we can assume that the center-of-mass motion of the atom is unaffected by the field, so that we can ignore (2.2.15). However, this is possible only because we are interested mainly in explaining laser action, which depends mostly on internal transitions within atoms or molecules, transitions based on the relative coordinate **x**. For other purposes Eq. (2.2.15) is essential. For example, the important topics of laser trapping and laser cooling depend directly on the effects produced by laser light on the atomic center of mass.

In order to proceed with (2.2.4) or (2.2.17) it is necessary to know $\mathbf{F}_{en}(\mathbf{x})$. For reasons that only quantum theory can explain (see Chapter 6), the classical theory satisfactorily treats many important features of the interaction of light with matter by adopting an *ad hoc* hypothesis about $\mathbf{F}_{en}$ due to H. A. Lorentz (around 1900). This hypothesis states that an electron in an atom responds to light as if it were bound to its atom or molecule by a simple spring. As a consequence the electron can be imagined to oscillate about the nucleus.

This *electron oscillator model* is often called the *Lorentz model* of an atom. It is not really a model of an atom as such, but rather a model of the way an atom responds to a perturbation. The Lorentz model was developed before atoms were understood to have massive nuclei. Lorentz simply asserted that each electron in an atom has a certain equilibrium position when there are no external forces. Under the influence of an electromagnetic field, the electron experiences the Lorentz force and is displaced from its equilibrium position, and according to Lorentz "the displacement will immediately give rise to a new force by which the particle is pulled back towards its original position, and which we may therefore appropriately distinguish by the name of *elastic force*."[2] Lorentz's assertion is equivalent to the replacement $\mathbf{F}_{en}(\mathbf{x}) \to -k_s\mathbf{x}$, where k_s is the "spring constant" associated with the hypothetical elastic force. This leads to the equation

$$m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E}(\mathbf{R}, t) - k_s\mathbf{x}$$

(Lorentz-model fundamental equation) (2.2.18a)

or

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right)\mathbf{x} = \frac{e}{m}\mathbf{E}(\mathbf{R}, t), \tag{2.2.18b}$$

where we have defined the electron's natural oscillation frequency $\omega_0 = \sqrt{k_s/m}$.

It is easy to see how a nonzero dipole moment density **P** such as is required in the wave equation (2.1.13) arises in the electron oscillator model. Let us continue to suppose that each atom of the medium has only one electron. When a field is applied, each atom's electron is displaced by some **x** from its original position (Figure 2.2). Thus, according to (2.2.14), each atom has a dipole moment $\mathbf{p} = e\mathbf{x}$. (The electron charge e is negative, so that **p** points from the negative charge to the positive.) If the density of atoms is denoted N, then the density of dipole moments is N times the individual dipole moment of each atom. That is, we have the polarization density

$$\mathbf{P} = N\mathbf{p} = Ne\mathbf{x} \tag{2.2.19}$$

induced in the medium by the field.

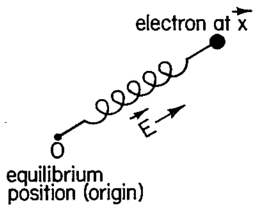

Figure 2.2 The electron oscillator (Lorentz) model of an atomic electron. An applied field displaces the electron from its equilibrium position. The atom reacts as though the electron were a charged mass on a spring.

This expression for **P** closes a circle. We now have a model for the interaction between electromagnetic radiation and a material medium consisting of one-electron atoms. The Maxwell equation (2.1.13) tells us how the electric field **E** depends upon the dipole moment density **P** of the medium. Newton's equation (2.2.18) tells us how the electron displacement **x** depends upon the electric field **E**. And Eq. (2.2.19) connects these basic equations by relating **P** to **x**. The electron oscillator model thus ties together the Maxwell equations with Newton's law of motion. Solutions of these coupled equations will provide the model's predictions about the mutual interaction of light and matter.

2.3 REFRACTIVE INDEX AND POLARIZABILITY

In many applications it is necessary to know the effect on a bound electron of a monochromatic field, i.e., a field varying purely sinusoidally in time with the single frequency ω. It will often be convenient for our purposes to consider a monochromatic *plane* wave, i.e., a monochromatic wave having the same value across a plane perpendicular to the direction (z, say) of propagation. We will furthermore deal initially with a linearly polarized wave. In this case the electric field at the position of the atom has the form

$$\mathbf{E}(z, t) = \hat{\boldsymbol{\varepsilon}}E_0\cos(\omega t - kz) \tag{2.3.1}$$

where k is an undetermined constant. We assume that the wave amplitude E_0 does not depend on t or z. The unit vector $\hat{\boldsymbol{\varepsilon}}$ specifies the polarization of the wave and lies in the xy plane, normal to the z axis (i.e., the wave is "transverse"). The period of the wave described by (2.3.1) is $2\pi/\omega$, and so the frequency ν is $\omega/2\pi$. Finally we recall that the wavelength, or spatial period, of the wave is $\lambda = 2\pi/k$.

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[2] H. A. Lorentz, *The Theory of Electrons* (Dover, New York, 1952), p. 9.

The electric field (2.3.1) is transverse, satisfying condition (2.1.12):

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = 0 \tag{2.3.2}$$

because it does not depend upon x or y and because $E_z = 0$ (because $\varepsilon_z = 0$). If $\mathbf{E}$ is also to be a solution of Maxwell's equations in *vacuum*, it must satisfy the wave equation (2.1.13) with $\mathbf{P} = 0$:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{2.3.3}$$

Suppose we try to satisfy this wave equation with the plane wave given in Eq. (2.3.1). Then since $\nabla^2 \mathbf{E} = \partial^2 \mathbf{E}/\partial z^2 = -k^2 \mathbf{E}$ and $\partial^2 \mathbf{E}/\partial t^2 = -\omega^2 \mathbf{E}$ in this case, we must have

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) \hat{\varepsilon} E_0 \cos(\omega t - kz) = 0 \tag{2.3.4}$$

Except in the trivial and physically uninteresting case $E_0 = 0$, we can satisfy this equation only if the constant k satisfies

$$k^2 = \frac{\omega^2}{c^2} \tag{2.3.5}$$

This relation between k and ω is an example of a dispersion relation. We have the two possibilities $k = \pm\, \omega/c$, or

$$\mathbf{E}(z, t) = \hat{\varepsilon} E_0 \cos \omega(t \pm z/c).$$

The choices $k = \omega/c$ or $k = -\omega/c$ correspond respectively to waves propagating in the positive or negative z direction with the phase velocity

$$v_p = \frac{\omega}{|k|} = c \tag{2.3.6}$$

This merely tells us something we already know: the propagation velocity of an electromagnetic wave in vacuum is c, the (vacuum) speed of light, and (2.3.5) is sometimes called the vacuum dispersion relation.

Now let us consider the case of a material medium of propagation, where the right-hand side of Eq. (2.1.13) is not zero. According to the electron oscillator model of a medium of one-electron atoms, $\mathbf{P}$ is given by Eq. (2.2.19), with the electron displacement $\mathbf{x}$ satisfying the equation of motion (2.2.18). If the electric field (2.3.1) is to be a solution of the coupled Maxwell–Newton equations it must be the driving field in the Newton equation (2.2.18):

$$\frac{d^2 \mathbf{x}}{dt^2} + \omega_0^2 \mathbf{x} = \hat{\varepsilon} \frac{e}{m} E_0 \cos(\omega t - kz) \tag{2.3.7}$$

This equation has the solution[3]

$$\mathbf{x} = \hat{\varepsilon} \left(\frac{eE_0/m}{\omega_0^2 - \omega^2} \right) \cos(\omega t - kz) \tag{2.3.8}$$

It is sometimes convenient to write this solution in the form

$$\mathbf{p} = e\mathbf{x} = \alpha \mathbf{E} \tag{2.3.9}$$

where we define the electronic *polarizability*

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2} \tag{2.3.10}$$

as the ratio of the induced dipole moment $\mathbf{p}$ of the atom to the electric field $\mathbf{E}$ that produces this dipole moment.

Thus we have the dipole moment density

$$\mathbf{P} = N\mathbf{p} = N\alpha(\omega)\, \mathbf{E} = \hat{\varepsilon} \left(\frac{Ne^2/m}{\omega_0^2 - \omega^2} \right) E_0 \cos(\omega t - kz) \tag{2.3.11}$$

This is the polarization predicted by the electron oscillator model when there is a field (2.3.1) in the medium. This solution for the polarization provides the source term on the right-hand side of the Maxwell equation (2.1.13). In order that we have a consistent theory, the electric field (2.3.1) appearing in (2.3.11) must satisfy (2.1.13). Using the assumed plane-wave, monochromatic solution (2.3.1) for $\mathbf{E}$ in (2.1.13), together with the above results for $\mathbf{P}$, we obtain the consistency condition

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) \hat{\varepsilon} E_0 \cos(\omega t - kz) = -\frac{\omega^2}{c^2} \frac{N\alpha(\omega)}{\epsilon_0} \hat{\varepsilon} E_0 \cos(\omega t - kz) \tag{2.3.12}$$

---

[3] This is the particular solution. The homogeneous solution has been omitted for reasons that are explained in Section 3.3.

To satisfy this equation k must satisfy a more general dispersion relation:

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{N\alpha(\omega)}{\epsilon_0}\right) = \frac{\omega^2}{c^2} n^2(\omega) \tag{2.3.13}$$

where we have defined

$$n^2(\omega) = 1 + \frac{N\alpha(\omega)}{\epsilon_0} \tag{2.3.14a}$$

or

$$n(\omega) = \left(1 + \frac{N\alpha(\omega)}{\epsilon_0}\right)^{1/2} = \left(1 + \frac{Ne^2/m\epsilon_0}{\omega_0^2 - \omega^2}\right)^{1/2} \tag{2.3.14b}$$

The plane monochromatic wave (2.3.1) is therefore a solution of the Maxwell equation (2.3.3) *in vacuum* if

$$k^2 = \frac{\omega^2}{c^2} \quad \text{(vacuum)} \tag{2.3.15}$$

but it can be a solution *in a material medium* only if

$$k^2 = n^2(\omega)\frac{\omega^2}{c^2} \quad \text{(material medium)} \tag{2.3.16}$$

In the second case the phase velocity of the wave is

$$v_p = \frac{\omega}{|k|} = \frac{c}{n(\omega)} \tag{2.3.17}$$

which identifies $n(\omega)$ as the refractive index of the medium for light of frequency ω.

The result (2.3.14) for the refractive index applies to a medium of one-electron atoms. This restriction may be removed. If we assume that the Z electrons in an atom respond independently of one another to an imposed field, then the displacement $\mathbf{x}_i$ of each electron satisfies (2.2.18) with perhaps a different "spring constant" k_i for each electron. The solution is given by (2.3.8):

$$\mathbf{x}_i = \hat{\boldsymbol{\varepsilon}}\left(\frac{eE_0/m}{\omega_i^2 - \omega^2}\right)\cos(\omega t - kz) \tag{2.3.18}$$

where $\omega_i = (k_i/m)^{1/2}$ is the natural oscillation frequency of the ith electron. The dipole moment (2.3.9) of the atom is replaced by

$$\mathbf{p} = \sum_{i=1}^{Z} e\mathbf{x}_i \tag{2.3.19}$$

where the summation is over the displacements of the Z electrons in the atom. From (2.3.9), therefore, we have the multielectron polarizability

$$\alpha(\omega) = \sum_{i=1}^{Z} \frac{e^2/m}{\omega_i^2 - \omega^2} \tag{2.3.20}$$

and the dipole density

$$\mathbf{P} = \hat{\boldsymbol{\varepsilon}} N\left(\sum_{i=1}^{Z} \frac{e^2/m}{\omega_i^2 - \omega^2}\right)E_0 \cos(\omega t - kz) \tag{2.3.21}$$

It follows that the generalization of (2.3.14) for a medium of Z-electron atoms is

$$n(\omega) = \left(1 + \frac{N}{\epsilon_0}\alpha(\omega)\right)^{1/2} = \left(1 + \frac{N}{\epsilon_0}\sum_{i=1}^{Z} \frac{e^2/m}{\omega_i^2 - \omega^2}\right)^{1/2} \tag{2.3.22a}$$

A further generalization occurs if more than one species of atom is present. If species a has Z_a electrons with resonant frequencies labeled ω_{ai}, then we have

$$n(\omega) = \left(1 + \sum_a \frac{N_a}{\epsilon_0}\sum_{i=1}^{Z_a} \frac{e^2/m}{\omega_{ai}^2 - \omega^2}\right)^{1/2} \tag{2.3.22b}$$

where N_a is the density of species a. In the case of gases, $n(\omega) \approx 1$ for optical frequencies and we can expand the square root in (2.3.22b) using the binomial formula $(1 + x)^m = 1 + mx + m(m - 1)x^2/2 + \cdots$. If we retain only the first-order correction to the zero-order result $n(\omega) = 1$, we find

$$n(\omega) = 1 + \sum_a \frac{N_a}{2\epsilon_0}\sum_{i=1}^{Z_a} \frac{e^2/m}{\omega_{ai}^2 - \omega^2} \tag{2.3.23}$$

This approximation implies that the index of a mixture of gases is the proportionate sum of their individual indices (see Problem 2.5).

TABLE 2.1 The Maxwell-Newton Equations of the Electron Oscillator Model

$$\nabla \cdot \mathbf{D} = 0, \qquad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{P} = Ne \sum_{i=1}^{Z} \mathbf{x}_i$$

$$\frac{d^2\mathbf{x}_i}{dt^2} + \omega_i^2 \mathbf{x}_i = \frac{e}{m} \mathbf{E}$$

Using Eqs. (2.3.1), (2.3.13), and (2.3.21), we easily obtain from the Maxwell equations the corresponding solutions for **B** and **D**. In Table 2.1 we collect the coupled Maxwell–Newton equations of the electron oscillator model, and in Table 2.2 we display the particular solutions we have found for these coupled equations. For definiteness we have taken the polarization unit vector $\hat{\boldsymbol{\varepsilon}}$ to be $\hat{\mathbf{x}}$, the unit vector in the x direction. The reader may easily verify that Table 2.2 provides a solution to the equations of Table 2.1.

It is sometimes useful to use the electric susceptibility $\chi(\omega)$ instead of the polarizability $\alpha(\omega)$. Since $\chi(\omega)$ is defined by the relation

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \tag{2.3.24}$$

the connection with $\alpha(\omega)$ is easily determined, by comparison with (2.3.11), to be

$$\chi(\omega) = N\alpha(\omega)/\epsilon_0 \tag{2.3.25}$$

Apart from the dimensional factor $1/\epsilon_0$ (which is not even present in cgs units) the only physical difference between χ and α is that χ is directly associated with a medium (proportional to N) and α is a single-oscillator characteristic. Expressions for quantities associated with an optical medium are usually somewhat simpler to express in terms of χ; for example, Eqs. (2.3.14) for the index of refraction can be rewritten as

$$n^2(\omega) = 1 + \chi(\omega) \tag{2.3.26a}$$

TABLE 2.2 A Solution of the Maxwell–Newton Equations

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos \omega\left(t \pm n(\omega)\frac{z}{c}\right), \qquad E_0 = \text{constant}$$

$$\mathbf{D} = n^2(\omega)\, \mathbf{E}$$

$$\mathbf{B} = \mp \frac{n(\omega)}{c} \hat{\mathbf{y}} E_0 \cos \omega\left(t \pm n(\omega)\frac{z}{c}\right)$$

$$\mathbf{P} = \epsilon_0 [n^2(\omega) - 1]\, \mathbf{E}$$

$$\mathbf{x}_i = \frac{e/m}{\omega_i^2 - \omega^2} \mathbf{E}$$

$$n(\omega) = \left(1 + \frac{N}{\epsilon_0} \sum_{i=1}^{Z} \frac{e^2/m}{\omega_i^2 - \omega^2}\right)^{1/2}$$

or

$$n(\omega) = [1 + \chi(\omega)]^{1/2} \tag{2.3.26b}$$

In the Lorentz model, $\chi(\omega)$ is given by

$$\chi(\omega) = \frac{Ne^2}{m\epsilon_0} \sum_{i=1}^{Z} \frac{1}{\omega_i^2 - \omega^2} \tag{2.3.27}$$

• The atoms or molecules of a medium do not form a continuum, but have empty spaces between them. As a result, there is a difference between the "mean" field satisfying the wave equation (2.1.13) and the actual field acting on a given atom. In most cases the only practical consequence of this difference is that we must modify the expression (2.3.22) for the refractive index to read

$$\frac{n^2(\omega) - 1}{n^2(\omega) + 2} = \frac{N\alpha(\omega)}{3\epsilon_0} \tag{2.3.28}$$

The origin of this "Lorentz–Lorenz relation" is discussed in many textbooks on electromagnetic theory. Note that when the refractive index is close to unity, so that $n^2(\omega) + 2 \approx 3$, the Lorentz–Lorenz relation reduces to (2.3.22) or (2.3.26). •

2.4 THE CAUCHY FORMULA

We will now compare the formula (2.3.22) for the refractive index with experimental data. Such data often show an increase of the refractive index with increasing frequency (decreasing wavelength). This is familiar from the way a glass prism disperses (separates) white light into its spectral components, with violet being deviated more than red. An example of such "normal" dispersion appears in Figure 2.3, which gives the refractive index of He gas over a range of wavelengths extending from the infrared to the ultraviolet.

To compare the dispersion formula (2.3.22) predicted by the electron oscillator model with experimental results, we first write it in terms of wavelength, $\lambda = 2\pi c/\omega$ (This is convenient because one normally measures the wavelength of optical radiation rather than the frequency.):

$$n^2(\lambda) = 1 + \frac{Ne^2}{4\pi^2\epsilon_0 mc^2}\sum_{i=1}^{Z}\frac{\lambda_i^2\lambda^2}{\lambda^2 - \lambda_i^2} \tag{2.4.1}$$

where λ_i is a wavelength associated with the natural oscillation frequency ω_i of the electron oscillator model by the definition

$$\lambda_i = \frac{2\pi c}{\omega_i} \tag{2.4.2}$$

It follows immediately from (2.4.1) that if the refractive index at wavelength λ exceeds unity, the sum must be positive; if $n(\lambda) < 1$, then the sum is negative. Most transparent materials we encounter daily (e.g., air, water, glass) have refractive indices greater than unity at visible wavelengths. We therefore conclude that in most transparent media the electrons have oscillation wavelengths λ_i that are less than optical wavelengths, which lie roughly between 4000 and 7000 Å, 400–700 nm. From Eq. (2.4.1), we not only have $n(\lambda) > 1$ but also $dn/d\lambda < 0$ for $\lambda_i < \lambda$, consistent with the frequent observation mentioned above. Of course, in order that $n(\lambda) > 1$ it is really only necessary that at least one of the λ_i must be less than λ; we are postulating here that all of the λ_i are less than visible wavelengths λ.

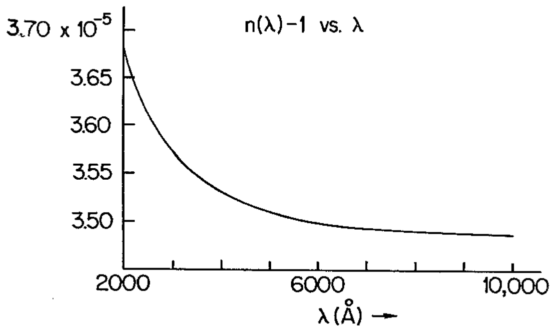

Figure 2.3 Refractive index of helium at standard temperature and pressure.

If $\lambda_i < \lambda$, we may again use the binomial series expansion

$$(1 + x)^m = 1 + mx + \tfrac{1}{2}m(m-1)x^2 + \cdots \tag{2.4.3}$$

which is valid for $|x| < 1$, to write

$$\frac{\lambda^2}{\lambda^2 - \lambda_i^2} = \left(1 - \frac{\lambda_i^2}{\lambda^2}\right)^{-1} = 1 + \frac{\lambda_i^2}{\lambda^2} + \left(\frac{\lambda_i^2}{\lambda^2}\right)^2 + \cdots \tag{2.4.4}$$

in Eq. (2.4.1). If λ_i^2/λ^2 is very small, i.e.,

$$\lambda_i^2 \ll \lambda^2 \tag{2.4.5}$$

we can approximate (2.4.4) by the first two terms of the expansion, and write

$$n^2(\lambda) \approx 1 + \frac{Ne^2}{4\pi^2\epsilon_0 mc^2}\sum_{i=1}^{Z}\lambda_i^2\left(1 + \frac{\lambda_i^2}{\lambda^2}\right) \tag{2.4.6}$$

as an approximation to (2.4.1). If we suppose further that

$$\left|n^2(\lambda) - 1\right| \ll 1 \tag{2.4.7}$$

we may again use (2.4.3) to obtain the approximation $(1 + x)^{1/2} \approx 1 + x/2$ for $x \ll 1$, and write the refractive index given by (2.4.6) as

$$n(\lambda) \approx 1 + \frac{Ne^2}{8\pi^2\epsilon_0 mc^2}\sum_{i=1}^{Z}\lambda_i^2\left(1 + \frac{\lambda_i^2}{\lambda^2}\right) \tag{2.4.8}$$

This is a good approximation to $n(\lambda)$ when the two conditions (2.4.5) and (2.4.7) are met.

Equation (2.4.8) explains why the refractive indices of many gases have long been known to follow the Cauchy formula

$$n(\lambda) - 1 = A\left(1 + \frac{B}{\lambda^2}\right) \tag{2.4.9}$$

where A and B are constants. Such an empirical relation was proposed by Cauchy in 1830, before the electromagnetic theory of light. For He gas at standard temperature and pressure (STP), for example, $A = 3.48 \times 10^{-5}$ and $B = 2.3 \times 10^{-11}$ cm^2. Now we notice that (2.4.8) is of precisely the form (2.4.9), with

$$A = \frac{Ne^2}{8\pi^2\epsilon_0 mc^2}\sum_{i=1}^{Z}\lambda_i^2 \tag{2.4.10}$$

and

$$B = \frac{\sum_{i=1}^{Z}\lambda_i^4}{\sum_{i=1}^{Z}\lambda_i^2} \tag{2.4.11}$$

This suggests that gases obeying the relation (2.4.9) satisfy the condition (2.4.5) assumed in the derivation of (2.4.8).

A more quantitative check of the electron oscillator model requires a knowledge of the natural wavelengths λ_i. The values of the λ_i are not given by the Lorentz oscillator model, since they are only a different manifestation of the hypothetical elastic force constants k_i. However, the λ_i can be determined empirically from dispersion data. Consider He gas at standard temperature and pressure. Helium is known to absorb strongly radiation of wavelength $\lambda \approx 584$ Å. Suppose we take this to be the natural wavelength λ_i of each electron in the electron oscillator model of He (atomic number $Z = 2$). Then

$$\sum_{i=1}^{Z}\lambda_i^2 = 2(584 \times 10^{-8}\ \text{cm})^2 = 6.82 \times 10^{-11}\ \text{cm}^2$$

$$\sum_{i=1}^{Z}\lambda_i^4 = 2(584 \times 10^{-8}\ \text{cm})^4 = 2.33 \times 10^{-21}\ \text{cm}^4$$

An ideal gas at STP has about 2.69×10^{19} atoms (or molecules) per cm^3 (Problem 2.7). Therefore we compute from (2.4.10) the value $A \approx 8.23 \times 10^{-5}$. This is more than twice the tabulated value, given above, that has been measured for He. For the coefficient B we get $\approx 3.42\ 10^{-11}$ cm^2, about 50% greater than the tabulated value.

This example brings out a general feature of our electron oscillator model: it is often in close *qualitative* accord with our observations but not in detailed quantitative agreement. Not infrequently it provides, as in this example, reasonable orders of magnitude. Before quantum mechanics, the electron oscillator model was modified *ad hoc* in order to bring its numerical predictions in line with experimental results. The nature of this modification will be discussed in Section 3.7, and in Chapter 6 we will interpret this modification from the standpoint of the quantum theory of the interaction of light with matter.

It is also interesting to consider the case in which the frequency of the radiation is much greater than the natural oscillation frequencies of the medium, i.e., the natural wavelengths are relatively very large: $\lambda_i >> \lambda$. The simplest example occurs for free electrons, for which there is no elastic binding force. The natural oscillation frequencies ω_i are then zero, and the dispersion formula (2.3.22) reduces to

$$n(\omega) = \left(1 - \frac{Ne^2}{\epsilon_0 m\omega^2}\right)^{1/2} \tag{2.4.12}$$

where N is now the density of free electrons. In some cases this result is known to be fairly accurate; it is applicable, for instance, to the upper atmosphere, where ultraviolet solar radiation produces free electrons by photoionization (Problem 2.8). Another example is the refraction of X-rays by glass. In this case the natural oscillation frequencies of the medium are not zero, but are much less than X-ray frequencies. In both of these examples the refractive index is less than one. Thus X-rays propagating in glass have a phase velocity greater than c.

- Equation (2.4.12) for free electrons shows that the refractive index can even be a pure *imaginary* number. This occurs whenever $\omega < \omega_p$, where

$$\omega_p = \left(\frac{Ne^2}{m\epsilon_0}\right)^{1/2} \tag{2.4.13}$$

is called the plasma frequency. In terms of the plasma frequency the refractive index (2.4.12) of a free-electron gas may be written as

$$n(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = \frac{1}{\omega}(\omega^2 - \omega_p^2)^{1/2} \tag{2.4.14}$$

To understand the physical significance of an imaginary refractive index, let us consider again the solutions of the Maxwell–Newton equations we have displayed in Table 2.2. To be specific, the solution corresponding to plane-wave propagation in the positive z direction is

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}E_0 \cos\omega\left(t - n(\omega)\frac{z}{c}\right) \tag{2.4.15}$$

This is not feasible if $n(\omega)$ is imaginary. Since the electric field must obviously be a real quantity we can instead write

$$\mathbf{E}(z, t) = \text{Re}\,[\hat{\mathbf{x}}E_0\, e^{-i\omega[t - n(\omega)z/c]}] \tag{2.4.16}$$

where Re means "real part of". This reduces to (2.4.15) when $n(\omega)$ is real. For the free-electron case $n(\omega)$ is given by (2.4.14) and we have therefore

$$\mathbf{E}(z, t) = \text{Re}\,[\hat{\mathbf{x}}E_0\, e^{-i\omega t}\, e^{i(\omega^2-\omega_p^2)^{1/2}z/c}] \tag{2.4.17}$$

When $\omega < \omega_p$ we have

$$(\omega^2 - \omega_p^2)^{1/2} = i(\omega_p^2 - \omega^2)^{1/2} = icb(\omega) \tag{2.4.18}$$

with $b(\omega) = (1/c)(\omega_p^2 - \omega^2)^{1/2} > 0$. Equation (2.4.17) then gives

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}E_0\, e^{-b(\omega)z} \cos \omega t \tag{2.4.19}$$

indicating that the electric field decreases exponentially with penetration depth in the medium. It follows from Table 2.2 that the magnetic field is attenuated in the same manner.

The field (2.4.19) is not a propagating wave. Thus when the field frequency ω is less than the plasma frequency ω_p, the free-electron gas will not support a propagating mode of the field. The field is therefore reflected. This applies, for instance, to the propagation of radio waves in the earth's atmosphere. High-frequency FM radio waves are not reflected by the ionosphere ($\omega > \omega_p$), whereas the lower-frequency AM waves are. AM radio broadcasts therefore reach more distant points on the earth's surface (see Problem 2.9). •

2.5 ELECTRIC DIPOLE RADIATION

According to Eq. (2.3.8), a monochromatic electric field forces an electron into oscillation at the frequency of the field. The field thus induces an oscillating dipole moment in an atom, and according to (2.1.13) this dipole can serve as the source of a new electric field. The electromagnetic field radiated by an oscillating electric dipole is discussed in textbooks on electromagnetic theory.[4] We will only discuss the most important results needed for our purposes. In the next section we will use these results to treat the scattering of light by atoms.

Suppose we have a time-dependent electric dipole moment

$$\mathbf{p} = \hat{\mathbf{x}}p(t) \tag{2.5.1}$$

where the unit vector $\hat{\mathbf{x}}$ specifies the fixed direction in which the dipole is assumed to point. Then at the position $\mathbf{r}$ measured from the dipole the electric and magnetic field vectors are[4]

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) = {} & \frac{1}{4\pi\epsilon_0}\left[3(\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{x}}\right]\left[\frac{1}{r^3}p\left(t - \frac{r}{c}\right) + \frac{1}{cr^2}\frac{d}{dt}p\left(t - \frac{r}{c}\right)\right] \\ & + \frac{1}{4\pi\epsilon_0}\left[(\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{x}}\right]\frac{1}{c^2r}\frac{d^2}{dt^2}p\left(t - \frac{r}{c}\right)\end{aligned} \tag{2.5.2a}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c}(\hat{\mathbf{x}}\times\hat{\mathbf{r}})\left[\frac{1}{cr^2}\frac{d}{dt}p\left(t - \frac{r}{c}\right) + \frac{1}{c^2r}\frac{d^2}{dt^2}p\left(t - \frac{r}{c}\right)\right] \tag{2.5.2b}$$

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[4] See, for example, R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1964), vol. 2, Section 21-4, or J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), or J. B. Marion, *Classicial Electromagnetic Radiation* (Academic Press, New York, 1965).

where the unit vector $\hat{\mathbf{r}}$ points from the dipole to $\mathbf{r}$, and r is the magnitude of $\mathbf{r}$ (i.e., $\mathbf{r} = \hat{\mathbf{r}}r$). Note that, owing to the finite velocity c of light, the electric and magnetic fields at $\mathbf{r}$ at time t depend upon the dipole moment at the earlier (retarded) time $t - r/c$.

The simplest example of (2.5.2) is the electrostatic case in which $p(t)$ is time-independent. In this case the fields (2.5.2) reduce to

$$\mathbf{E}(\mathbf{r}, t) = \left[3(\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{x}}\right]\frac{p}{4\pi\epsilon_0 r^3} \tag{2.5.3a}$$

$$\mathbf{B}(\mathbf{r}, t) = 0 \tag{2.5.3b}$$

The reader may recall that these electrostatic dipole fields are derivable independently of the general formulas (2.5.2), simply by considering the static (Coulomb) fields of two point charges q and $-q$ separated by a distance d small compared with r; the electric dipole moment is $\mathbf{p} = q\mathbf{d}$ and the magnetic dipole moment is zero. However, we are interested now in the field (2.5.2) radiated by a *time-dependent* dipole moment, specifically the dipole moment induced in an atom by a time-dependent external field.

In particular, we are interested in the rate at which an oscillating electric dipole radiates electromagnetic energy. We will calculate this rate from the fields (2.5.2). To do this we use the *Poynting vector* of an electromagnetic field (recall Problem 2.6):

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \epsilon_0 c^2\mathbf{E}\times\mathbf{B} \tag{2.5.4}$$

The Poynting vector of the field (2.5.2) gives the rate at which electromagnetic energy crosses a unit area. Its SI units are W/m^2 but W/cm^2 is more commonly used in laser physics. The total rate at which electromagnetic energy is radiated may be obtained by integrating the normal (radially outward) component of $\mathbf{S}$ over a large sphere of radius R centered at the dipole. Since the differential element of area on a sphere is given in terms of the polar angle θ and the azimuthal angle ϕ by

$$dA = R^2 \sin\theta\, d\theta\, d\phi \tag{2.5.5}$$

the radiated power (written Pwr) at the spherical surface is

$$\text{Pwr} = R^2\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, \mathbf{S}(R, \theta, \phi)\cdot\hat{\mathbf{R}} \tag{2.5.6}$$

From (2.5.2) and the definition (2.5.4) of the Poynting vector, we see that $\mathbf{S}(R, \theta, \phi)$ has a term varying as R^{-2}, and other terms falling off as R^{-3}, R^{-4}, and R^{-5}. The terms decreasing faster than R^{-2} (so-called near-zone and inductionzone fields) do not contribute to outward power flow at any value of R, since they have zero time average. [In many texts it is pointed out that these terms make no contribution to the net energy flow at infinity, obtained by integrating the Poynting flux over a large sphere of radius R_0 surrounding the dipole:

$$\int_{R_0} \mathbf{S} \cdot \mathbf{n}\, d(\text{area}) = 0,$$

in the limit $R_0 \to \infty$. This is also true.] Only the term in $\mathbf{S}$ falling off as R^{-2} contributes to the net rate of electromagnetic energy flow. This term arises from the so-called "radiation zone" fields:

$$\mathbf{E}(\mathbf{R}, t) = \frac{1}{4\pi\epsilon_0 c^2 R}[(\hat{\mathbf{x}} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \hat{\mathbf{x}}]\frac{d^2}{dt^2}p\left(t - \frac{R}{c}\right) \tag{2.5.7a}$$

$$\mathbf{B}(\mathbf{R}, t) = \frac{1}{4\pi\epsilon_0 c^3 R}[\hat{\mathbf{x}} \times \hat{\mathbf{R}}]\frac{d^2}{dt^2}p\left(t - \frac{R}{c}\right) \tag{2.5.7b}$$

which follow from (2.5.2) when R is so large that

$$|p| \ll \frac{R}{c}\left|\frac{dp}{dt}\right| \quad \text{and} \quad \left|\frac{dp}{dt}\right| \ll \frac{R}{c}\left|\frac{d^2p}{dt^2}\right| \tag{2.5.8}$$

In the radiation zone the electric and magnetic fields (2.5.7) are orthogonal to each other and to $\hat{\mathbf{R}}$.

The Poynting vector in the radiation zone defined by (2.5.8) follows from (2.5.4) and (2.5.7):

$$\mathbf{S} = \frac{\epsilon_0 c}{(4\pi\epsilon_0)^2}\left(\frac{1}{c^2 R}\right)^2 [(\hat{\mathbf{x}} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \hat{\mathbf{x}}] \times [\hat{\mathbf{x}} \times \hat{\mathbf{R}}] \times \left[\frac{d^2}{dt^2}p\left(t - \frac{R}{c}\right)\right]^2 \tag{2.5.9}$$

The vector cross product in this equation may be evaluated using the identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{2.5.10}$$

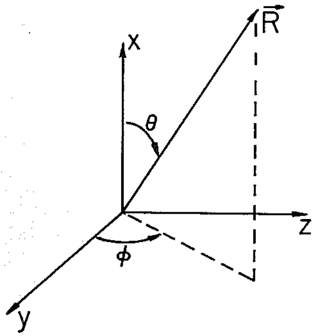

Figure 2.4 The polar angle θ in Eqs. (2.5.11) and (2.5.12) is measured from the dipole axis.

to obtain

$$\begin{aligned}[(\mathbf{x} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \hat{\mathbf{x}}] \times [\hat{\mathbf{x}} \times \hat{\mathbf{R}}] &= [(\hat{\mathbf{x}} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} \times (\hat{\mathbf{x}} \times \hat{\mathbf{R}}) - \hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \hat{\mathbf{R}})] \\ &= \hat{\mathbf{R}}[1 - (\hat{\mathbf{R}} \cdot \hat{\mathbf{x}})^2] \\ &= \hat{\mathbf{R}} \sin^2\theta \end{aligned} \tag{2.5.11}$$

where θ is the angle between the dipole axis and the radius vector $\mathbf{R}$. (Figure 2.4). Thus

$$\mathbf{S} = \hat{\mathbf{R}}\left(\frac{\sin^2\theta}{16\pi^2\epsilon_0 c^3 R^2}\right)\left[\frac{d^2}{dt^2}p\left(t - \frac{R}{c}\right)\right]^2 \tag{2.5.12}$$

which has the angular dependence (or "radiation pattern") depicted in Figure 2.5. The radiation is greatest in directions normal to the dipole axis ($\theta = \pi/2$), and falls to zero in the two directions parallel to the axis ($\theta = 0, \pi$).

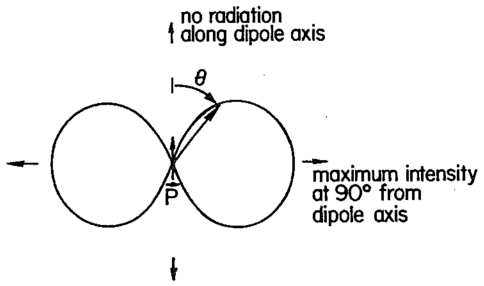

Figure 2.5 The radiation pattern ($\sin^2\theta$) of a linear electric dipole oscillator. The three-dimensional pattern is obtained by rotating the two-lobe pattern in the figure about the dipole (vertical) axis.

The total rate at which electromagnetic energy flows out from the dipole at radius R may now be calculated using (2.5.6) and (2.5.12):

$$\begin{aligned}
\text{Pwr} &= R^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, \hat{\mathbf{R}} \cdot \mathbf{S} \\
&= R^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left(\frac{\sin^2\theta}{16\pi^2\epsilon_0 c^3 R^2}\right)\left[\frac{d^2}{dt^2} p\left(t - \frac{R}{c}\right)\right]^2 \\
&= \frac{1}{16\pi^2\epsilon_0 c^3}\left[\frac{d^2}{dt^2} p\left(t - \frac{R}{c}\right)\right]^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta \\
&= \left(\frac{1}{4\pi\epsilon_0}\right)\frac{2}{3c^3}\left[\frac{d^2}{dt^2} p\left(t - \frac{R}{c}\right)\right]^2 \qquad (2.5.13)
\end{aligned}$$

The rate at which the dipole at the center of our fictitious sphere radiates electromagnetic energy is therefore

$$\text{Pwr} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{2}{3c^3}\left(\frac{d^2 p}{dt^2}\right)^2 \qquad (2.5.14)$$

• This result is similar to a general (nonrelativistic) formula for the power radiated by a particle of charge q and acceleration $\mathbf{a}$,

$$\text{Pwr} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{2q^2a^2}{3c^3} \qquad (2.5.15)$$

From this formula an estimate can be made of the length of time an oscillator *not* driven by an external field, but simply freely oscillating, can be expected to take to convert its oscillation energy to radiation.

If the radiative "lifetime" of an oscillator obtained in this way is designated τ, then $1/\tau$ is the radiation rate, and one finds (see Problem 2.10)

$$\frac{1}{\tau} = \frac{1}{4\pi\epsilon_0}\frac{2e^2\omega_0^2}{3mc^3} \qquad (2.5.16)$$

It is interesting that the ad hoc replacement of half the oscillator's energy (the potential energy) by half the lowest energy of a quantum oscillator (see Section 4.5) turns (2.5.16) into a quantum-mechanical formula for the spontaneous emission rate. That is, if we use $\frac{1}{2}m\omega_0^2x^2 = \frac{1}{2}(\hbar\omega_0/2)$ to eliminate m from (2.5.16), we find

$$\frac{1}{\tau} = \frac{1}{4\pi\epsilon_0}\frac{4e^2x^2\omega_0^3}{3\hbar c^3} \qquad (2.5.17)$$

This is the correct quantum-mechanical rate for spontaneous radiative decay, if the oscillator amplitude x is interpreted appropriately as a coordinate "matrix element" (see Eq. (7.6.5)) •

2.6 RAYLEIGH SCATTERING

Using the solutions of Table 2.2, we may write the dipole moment induced in an atom by a monochromatic plane wave traveling in the z direction as

$$\mathbf{p} = e\sum_{i=1}^{Z}\mathbf{x}_i = \frac{\mathbf{P}}{N} = \hat{\mathbf{x}}\alpha(\omega)E_0\cos\omega\left(t - \frac{z}{c}\right) \qquad (2.6.1)$$

where $\alpha(\omega)$ is the atomic dipole polarizability, given in (2.3.20). We assume the atom to be in a dilute host medium such as a gas for which $n(\omega) = 1$ is a satisfactory approximation. We will now apply the results we have obtained for electric dipole radiation to the case where $\mathbf{p}$ is given by (2.6.1). That is, we will calculate the power radiated by any atom as a result of its having an oscillating dipole moment induced by a monochromatic field.

For the dipole moment (2.6.1) we have

$$\frac{d^2\mathbf{p}}{dt^2} = -\omega^2\alpha(\omega)\hat{\mathbf{x}}E_0\cos\omega\left(t - \frac{z}{c}\right) \qquad (2.6.2)$$

According to (2.5.14), therefore, electromagnetic energy is radiated at the rate

$$\text{Pwr} = \frac{\omega^4}{6\pi\epsilon_0 c^3}\alpha^2(\omega)E_0^2\cos^2\omega\left(t - \frac{z}{c}\right) \qquad (2.6.3)$$

This is the instantaneous power radiated by a single dipole. It has been obtained by applying the general result (2.5.14) of electromagnetic theory to the solutions of the coupled Maxwell–Newton equations given in Table 2.2.

Using the fields $\mathbf{E}$ and $\mathbf{B}$ of Table 2.2 in (2.5.4), we obtain the Poynting vector [recall in this case we have assumed $n(\omega) = 1$]

$$\mathbf{S} = \epsilon_0 c\, \hat{\mathbf{z}}E_0^2\cos^2\omega\left(t - \frac{z}{c}\right) \qquad (2.6.4a)$$

Therefore

$$E_0^2\cos^2\omega\left(t - \frac{z}{c}\right) = \frac{1}{\epsilon_0 c}|\mathbf{S}| \qquad (2.6.4b)$$

and so the power (2.6.3) radiated by the dipole may be written as

$$\mathrm{Pwr} = \frac{8\pi\omega^4/c^4}{3(4\pi\epsilon_0)^2}\,\alpha^2(\omega)\,|\mathbf{S}| \tag{2.6.5}$$

The radiation from an atom having an oscillating electric dipole moment goes off in all directions in accordance with the angular distribution (2.5.12). This means there is a spatial redistribution or *scattering* of electromagnetic radiation. The incident plane wave still travels in the z direction, but with diminished amplitude because of the scattering. The scattering of radiation out of the incident wave is indicated pictorially in Figure 2.6.

Equation (2.6.5) shows that the power radiated by an atom is linearly proportional to $|\mathbf{S}|$, the rate per unit area at which the field driving the dipole transports electromagnetic energy. Since Pwr has the dimensions of energy per unit time, the constant of proportionality in (2.6.5) has the dimensions of an area. This area is called the *scattering cross section*, and denoted $\sigma(\omega)$. It is given by

$$\sigma(\omega) = \frac{8\pi}{3}\omega^4\left(\frac{\alpha(\omega)}{4\pi\epsilon_0 c^2}\right)^2 \tag{2.6.6}$$

Note that cross section, like polarizability, is a single-atom parameter. If there is a collection of scattering dipoles, making up a continuous medium, then the cross section can be expressed in terms of the medium's index of refraction by the use of (2.3.22): $\epsilon_0[n^2(\omega) - 1] = N\alpha(\omega)$. Then (2.6.6) becomes

$$\sigma(\omega) = \frac{8\pi\omega^4}{3c^4}\left(\frac{n^2(\omega) - 1}{4\pi N}\right)^2 \tag{2.6.7}$$

The relation (radiated power) = (scattering cross section) × (field power per unit area incident on the dipole) means that the radiated power is the same as if the scattering atom (dipole) had a cross-sectional area σ, and *all* of the incident field power intercepted by this area were scattered. There is no actual geometrical object the same size as this cross section; it is merely a convenient measure of the effectiveness of the scatterer.

By scattering the incident light, the dipoles cause the light beam to be attenuated as it propagates. The degree of attenuation can be calculated as follows. Suppose there are N atoms (scatterers) per unit volume. According to (2.6.5) and (2.6.6), each one radiates electromagnetic energy at the rate σI, where the intensity I (typically measured in W/cm^2) of the incident wave is defined as the cycle-averaged magnitude of the Poynting vector:

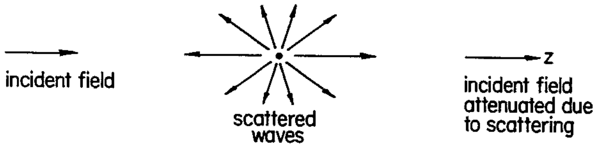

Figure 2.6 An incident field induces an oscillating dipole moment in an atom, which then radiates. The dipole radiation field is the field scattered by the atom.

$$I(t, z) = \frac{1}{T}\int_t^{t+T} |\mathbf{S}(t', z)|\, dt' \tag{2.6.8a}$$

where $T = 2\pi/\omega$. Therefore, in the case of a monochromatic plane wave $I(t, z) \rightarrow I(z)$. From (2.6.4) we find this intensity to be given by

$$I = \tfrac{1}{2}\,\epsilon_0 c |E_0|^2 \tag{2.6.8b}$$

Consider, as in Figure 2.7, an imaginary slab at position z in the medium, drawn so that it is perpendicular to the direction z of propagation of the wave. Let the energy density of the wave at the entrance face of the slab be denoted $u(z)$. As a result of scattering of electromagnetic radiation into directions of propagation other than z, the energy density $u(z + \Delta z)$ of the wave at the exit face of our imaginary slab must be less than $u(z)$. We may write

$$u(z + \Delta z) - u(z) \approx -N[\sigma(\omega)I(z)]\,(\Delta z/c) \tag{2.6.9}$$

The right-hand side of this expression is simply the number of atoms per unit volume, times the power scattered by each atom, times the time $\Delta t = \Delta z/c$ taken by the wave to cross the slab drawn in Figure 2.7. The approximation (2.6.9) is valid provided Δz is small enough that I remains relatively constant over the width of the slab, so that the same radiated power $\sigma I(z)$ may be associated with every atom in the slab. We divide by Δz in (2.6.9) and then take the limit $\Delta z \rightarrow 0$ to ensure the validity of this approximation. We obtain the differential equation

$$\frac{du}{dz} = -\frac{N\sigma(\omega)\,I(z)}{c} \tag{2.6.10}$$

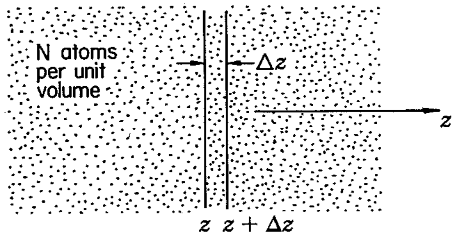

Figure 2.7 An imaginary thin slab of width Δz in the scattering medium.

where as usual

$$\frac{du}{dz} = \lim_{\Delta z \to 0} \frac{u(z + \Delta z) - u(z)}{\Delta z} \tag{2.6.11}$$

In many contexts in physics it is helpful to recognize that intensity I is simply energy density u times wave propagation velocity. This can be checked in the electromagnetic case by using the solutions in Table 2.2 to compute the energy density u [see Problem 2.6] for comparison with the intensity (magnitude of Poynting vector) obtained in (2.6.4). The result for the case $n = 1$ is

$$I(z) = cu(z) \tag{2.6.12}$$

A simple analogy to this result is a stream of pellets moving with speed v. If ρ is the number of pellets per unit volume, then the "intensity," or flux, of the stream is

$$\Phi = \rho v \tag{2.6.13}$$

This is the number of pellets crossing a unit area per unit time. In Eq. (2.6.12) the intensity I and energy density u of the electromagnetic wave play the roles of Φ and ρ, respectively, and the light velocity c plays the role of v.

It follows from (2.6.10) and (2.6.12) that

$$\frac{dI}{dz} = -N\sigma(\omega)\, I(z) \tag{2.6.14}$$

The solution to this simple differential equation is

$$I(z) = I_0\, e^{-a_s(\omega)z} \tag{2.6.15}$$

This gives the intensity at a distance z into the medium, given the "initial condition" that the intensity at $z = 0$ is I_0. The initial condition may refer to the actual entrance to the scattering medium, or merely some convenient reference plane inside the medium. We have defined

$$a_s(\omega) = N\sigma(\omega) \tag{2.6.16}$$

which is called the *extinction coefficient due to scattering*. The length a_s^{-1} is the distance of propagation for which the intensity decreases by a factor e^{-1} as a result of scattering. The relation (2.6.16) provides a general connection between the density N of scatterers, the cross section σ associated with each scatterer, and the extinction coefficient due to scattering. An expression of the same form as (2.6.15) also describes the attenuation of the flux of a beam of particles in a medium of scatterers.

Using (2.6.7) and (2.6.16), we may conveniently express the extinction coefficient in terms of the refractive index:

$$a_s(\omega) = \left(\frac{\omega}{c}\right)^4 \frac{[n^2(\omega) - 1]^2}{6\pi N} \tag{2.6.17}$$

Equation (2.6.17) was derived by Lord Rayleigh (J.W. Strutt) in 1899. Rayleigh scattering is characterized by the extinction coefficient (2.6.17).

• There is a conceptual inconsistency in our (and most textbooks') treatment of attenuation here and our solution of the Maxwell–Newton equations earlier. Recall that in Table 2.2 we have given the fully exact solutions for the electromagnetic fields **E** and **B** propagating through a continuous medium of atomic dipoles characterized by a real index of refraction $n(\omega)$. The solutions given there show no evidence of scattered light or of attenuation of the incident beam. This is correct. In fact there is no attenuation when a light beam passes through such an ideal collection of dipoles.

However, an important assumption of our idealized derivation is usually violated for atmospheric light propagation. This assumption is that the density of scatterers N is constant. If the density were actually constant, the scattered fields from individual atoms or molecules would actually cancel out in every direction except the forward z direction, with the solutions of Table 2.2 as the result.

When N is not constant in either space or time the side-scattered fields do not cancel out, and appreciable side scattering can occur, as Rayleigh's formula predicts. However, since the assumption of constant N was used in Rayleigh's derivation [and in ours in the substitution between (2.6.6) and (2.6.7)], that leaves open the matter of explaining why (2.6.17) is correct. This is a significant problem of statistical physics, first solved by Smoluchowski and Einstein in the first decade of the century, which we need not discuss further here. •

The frequencies of the scattered and incident fields are equal in Rayleigh scattering. This is an inevitable consequence of the electron oscillator model. The induced dipole moment oscillates at the frequency of the driving (incident) field, and therefore radiates at this frequency. Indeed, this feature of Rayleigh scattering is predicted by *any* model in which the induced dipole moment is linearly proportional to the imposed field.

The ω^4 (i.e., $1/\lambda^4$) dependence of the extinction coefficient (2.6.17) for Rayleigh scattering means that the amount of scattering increases sharply with increasing frequency. (The refractive index n generally varies much more slowly with ω than ω^4.) Rayleigh used (2.6.17) to explain why the sky is blue and the sunset is red. When we look at the sky away from the sun on a sunny day, we see light that has been scattered by air molecules exposed to sunlight. This scattered light is predominantly blue because the high-frequency components of the visible solar radiation are scattered more strongly than the low-frequency components. The sunset, however, is reddish because the sunlight has traveled a sufficient distance through the earth's atmosphere that much of the high-frequency component has been scattered away.

Consider the Rayleigh scattering of visible radiation by molecules in the earth's atmosphere. Taking $\lambda = 6000$ Å, and $n \approx 1.0003$ for the refractive index of air at optical frequencies, we find from (2.6.17) that

$$a_s^{-1} \approx 4.4 \times 10^{-13} N \text{ cm} \tag{2.6.18}$$

where N is the number of air molecules in a cubic centimeter. Assuming an ideal gas at STP, we have $N \approx 2.69 \times 10^{19}$ (Problem 2.7) and therefore

$$a_s^{-1} \approx 118 \text{ km} \tag{2.6.19}$$

for the distance in which 6000-Å radiation is attenuated by a factor $e^{-1} \approx 37\%$ at sea level. Rayleigh compared such calculations with astronomers' estimates for the transmission of stellar radiation through the earth's atmosphere. He drew the important conclusion that scattering of light by molecules alone, without suspended particles (dust), is strong enough to cause the blue sky, which he poetically called the "heavenly azure" (see also Problem 2.12).

This explanation of the blue sky suggests, in fact, that the sky should be violet, since violet light should be scattered more strongly than blue. One reason the sky appears blue rather than violet is that the eye is more sensitive to blue. Furthermore the solar spectrum is not uniform, but has somewhat less radiation at the shorter visible wavelengths.

2.7 POLARIZATION BY RAYLEIGH SCATTERING

A less obvious characteristic of skylight (in contrast to direct sunlight) is that it is polarized. This effect, which was observed in 1811 by D.F. Arago, is easily observed with Polaroid sunglasses. The extent of polarization appears to be strongest from directions near 90° to the direction of the sun from the observer. It has been observed that bees and certain other insects are sensitive to the polarization of light and use it for navigation. Human eyes, of course, are not directly sensitive to polarization.

To understand the polarization of light by Rayleigh scattering, let us return again to the electric field (2.5.7a) in the radiation zone of an oscillating dipole. The unit vector $\hat{\mathbf{x}}$, defining the direction of the dipole moment, also defines the polarization of the incident field (propagating in the z direction). $\hat{\mathbf{R}}$ is the unit vector pointing from the dipole to the point of observation. If we observe the scattered field at right angles to the "plane of incidence" defined by the directions of polarization and propagation, we see from (2.5.7a) that it will be polarized in the x direction, since

$$(\hat{\mathbf{x}} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \hat{\mathbf{x}} = (\hat{\mathbf{x}} \cdot \hat{\mathbf{y}})\hat{\mathbf{y}} - \hat{\mathbf{x}} = -\hat{\mathbf{x}} \tag{2.7.1}$$

when $\hat{\mathbf{R}} = \hat{\mathbf{y}}$ (Figure 2.8). If instead the incident field driving the dipole is polarized in the y direction, there is no scattered field in the y direction, since

$$(\hat{\mathbf{y}} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \hat{\mathbf{y}} = (\hat{\mathbf{y}} \cdot \hat{\mathbf{y}})\hat{\mathbf{y}} - \hat{\mathbf{y}} = 0 \tag{2.7.2}$$

The light from the sun is unpolarized. In fact virtually all nonlaser sources, including fluorescent and incandescent lamps, emit unpolarized light. In such sources, thermal energy is converted via collisions into internal atomic energy. According to the electron oscillator model, this internal energy appears as electron oscillations. We may imagine a thermal source of radiation to be a large collection of dipole oscillators (atoms) which each radiate, independently of one another, the power (2.5.14). At any instant the dipole moment of each atom oscillates along some axis, thus radiating a field with a polarization determined by this axis [Eq. (2.5.7a)]. The total electric field at any frequency is the sum of the fields from the individual atoms. This (total) radiated field at a given point has a direction determined by the vector addition of the individual atoms' fields. However, this direction varies rapidly in time. Each individual atom radiates for only a short time, typically about ten nanoseconds, before losing the internal energy picked up in a collision (Problem 2.10). After such an interval a different subset of atoms contributes to the total field, which will then point in a different direction. The field from such a source is "unpolarized" because we cannot detect the rapid variations of the electric field direction.

The direction of the dipole moment driven by an unpolarized wave propagating in the z direction (Figure 2.8) will therefore be varying rapidly in the xy plane. Equations (2.7.1) and (2.7.2) show that the dipole radiates in the y direction only when its oscillation has a nonzero component in the x direction (Figure 2.8), and

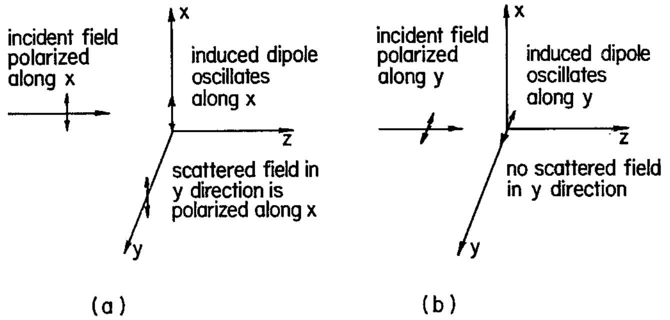

Figure 2.8 An incident field propagating in the z direction with polarization (*a*) along x, in which case the field scattered in the y direction is also polarized along x, and (*b*) along y, in which case there is no scattering in the y direction.

in that case the radiation is polarized in the x direction. Rayleigh scattering thus produces polarized light. This explains the polarization of skylight.

The theory of scattering becomes much more complicated when the particle dimensions are not negligible compared with the wavelength. In this case the light scattered at 90° is not completely polarized. And as the particle size increases, the scattering cross section becomes less sensitive to the wavelength; the radiation scattered from white light becomes "whiter" as the particle size increases. This explains why clouds, consisting of water droplets suspended in air, are white.

2.8 THE TYNDALL–RAYLEIGH EXPERIMENT

All of these features of light scattering may be observed beautifully in a simple experiment that the reader can perform with readily available materials (Figure 2.9). A few spoonfuls of photographic fixing powder (sodium thiosulfate) are dissolved in a beaker or small tank of water. The addition of about 100 ml of dilute sulfuric acid causes small grains of sulfur to precipitate out of solution after a few minutes. In the initial stages of precipitation these grains are very small. The scattered light has a faintly bluish hue, gradually becoming a deeper blue (the "blue sky"). The scattered light viewed at 90° (Figure 2.9) is observed to be strongly polarized. The light transmitted through the tank has a yellowish and eventually a strongly reddish hue (the "sunset"). After several more minutes the scattered light is not blue but white ("clouds"), and it is no longer strongly polarized. At this stage we are observing light scattering from sulfur grains that have grown to a size comparable to or larger than optical wavelengths.

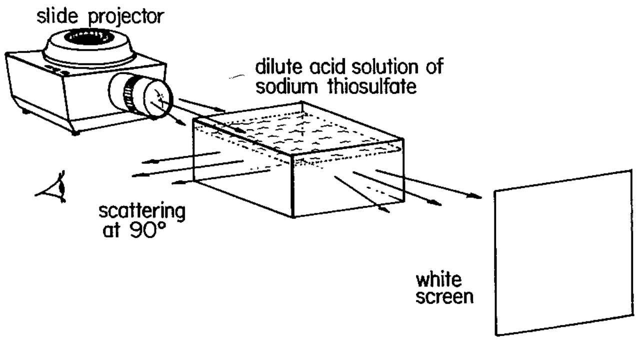

Figure 2.9 Experiment demonstrating the effect of particle size in light scattering. In the initial stage of precipitation the sulfur particles suspended in sodium thiosulfate solution are small compared with an optical wavelength. The tank takes on a blue color, the light scattered at 90° is strongly polarized, and on the white screen one observes the "red sunset." After a few minutes the particles have grown larger than a wavelength. Then the tank has a cloudy appearance and the light scattered at 90° is no longer strongly polarized.

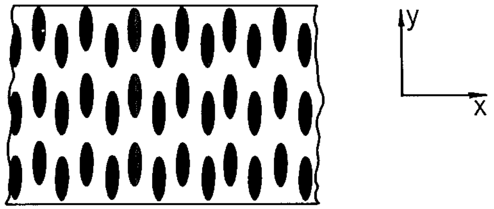

Figure 2.10 A birefringent material in which the molecules are aligned along the y axis.

Such an experiment was performed by John Tyndall in 1869. Rayleigh (1881) found that the "hypo" (sodium thiosulfate) solution demonstrates the scattering effects especially well.

It should be borne in mind, of course, that the notion of "color" is a subjective one. The theory of color vision has attracted the attention of many physicists, including Isaac Newton, Thomas Young, H. Helmholtz, J.C. Maxwell, Rayleigh, and E.H. Land.[5] Even now, however, color vision is not completely understood. Although it would be inappropriate for us to explore the subject here, it may be worthwhile to mention that in dim light we see nearly in "black and white." This explains, for example, why we do not see the brilliant colors of the Crab nebula that are apparent in photographs taken with long exposure times. Another example is the night sky itself. It looks black instead of blue because of its low intensity; a photograph taken with a long exposure time shows it to be "really" blue.

2.9 BIREFRINGENCE

We have tacitly assumed that the elastic restoring force acting on a bound electron is independent of the electron displacement $\mathbf{x}$. In general, however, the "spring constant" k_s may depend on the direction of $\mathbf{x}$, in which case the medium is said to be optically anisotropic. In such a medium the refractive index will be different for different polarizations of the field propagating in the medium. This is called *birefringence*. All transparent crystals with noncubic lattice structure, such as calcite, quartz, ice, and sugar, are birefringent.

It is easy to understand how such anisotropy might arise. In particular, imagine a long, rod-shaped molecule in which the elastic restoring force is different for electron displacements parallel and perpendicular to the axis. There will be different refractive indices for different directions of field polarization in a medium in which there is some degree of molecular alignment.

Consider a birefringent material in which the molecules are aligned along the y axis, as illustrated in Figure 2.10. This preferred direction may be called the *optic axis*. Waves that are linearly polarized in the x or y directions will propagate with different refractive indices n_x or n_y, respectively. That is, a field

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[5] E.H. Land, Experiments in Color Vision, *Scientific American*, **200**, May 1959, p. 84.

$$\mathbf{E}(z,t) = \hat{\boldsymbol{\varepsilon}} E_0 \cos(\omega t - kz)$$
$$= \hat{\boldsymbol{\varepsilon}} E_0 \cos \omega\left(t - \frac{nz}{c}\right) \tag{2.9.1}$$

in the medium will propagate with refractive index $n = n_x$ if $\hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{x}}$, or $n = n_y$ if $\hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{y}}$. For a general linearly polarized wave in which

$$\hat{\boldsymbol{\varepsilon}} = \epsilon_x \hat{\mathbf{x}} + \epsilon_y \hat{\mathbf{y}} \tag{2.9.2}$$

and $\epsilon_x^2 + \epsilon_y^2 = 1$, we have

$$\mathbf{E}(z,t) = \epsilon_x \hat{\mathbf{x}} E_0 \cos \omega\left(t - \frac{n_x z}{c}\right) + \epsilon_y \hat{\mathbf{y}} E_0 \cos \omega\left(t - \frac{n_y z}{c}\right) \tag{2.9.3}$$

If the field

$$\mathbf{E}(0,t) = \epsilon_x \hat{\mathbf{x}} E_0 \cos \omega t + \epsilon_y \hat{\mathbf{y}} E_0 \cos \omega t \tag{2.9.4}$$

is incident at the face $z = 0$ of the material, the field at $z = l$ in the material will be

$$\mathbf{E}(l,t) = \epsilon_x \hat{\mathbf{x}} E_0 \cos \omega\left(t - \frac{n_x l}{c}\right) + \epsilon_y \hat{\mathbf{y}} E_0 \cos \omega\left(t - \frac{n_y l}{c}\right) \tag{2.9.5}$$

Thus the x and y polarization components of the field, since they propagate with different phase velocities, develop a phase difference

$$\phi(l) = \frac{\omega l}{c}(n_y - n_x) \tag{2.9.6}$$

Therefore we may write (2.9.5) as

$$\mathbf{E}(l,t) = \epsilon_x \hat{\mathbf{x}} E_0 \cos(\omega t - k_x l) + \epsilon_y \hat{\mathbf{y}} E_0 \cos[\omega t - k_x l - \phi(l)] \tag{2.9.7}$$

where $k_x = n_x \omega / c$ and $\phi(l)$ [Eq. (2.9.6)] is the phase difference of the x and y components after a propagation distance l.

This phase difference is used, for instance, in the construction of a *quarter-wave plate*. A quarter-wave plate is a sheet of birefringent material of just the right thickness l that $|\phi(l)| = \pi/2$, i.e., the phase difference between waves linearly polarized parallel and perpendicular to the optic axis is a quarter of a cycle. In other words, the optical path difference $l|n_y - n_x|$ is a quarter of the wavelength under consideration. If light incident on a quarter-wave plate is linearly polarized at 45° to the optic axis, so that $|\epsilon_x| = |\epsilon_y|$, then the transmitted field has the form [Eq. (2.9.7)]

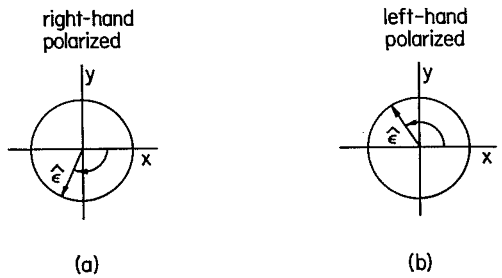

Figure 2.11 The x and y components of a circularly polarized field propagating in the z direction trace out a circle: (*a*) right-hand circular polarization; (*b*) left-hand circular polarization.

$$\mathbf{E}(l,t) = \frac{1}{\sqrt{2}} E_0 [\hat{\mathbf{x}} \cos(\omega t - k_x l) \pm \hat{\mathbf{y}} \sin(\omega t - k_x l)] \tag{2.9.8}$$

since $|\epsilon_x| = |\epsilon_y| = 1/\sqrt{2}$. The x and y components of this electric field trace out a circle, as indicated in Figures 2.11 *a* and *b*. In other words, the transmitted field is circularly polarized. It is called right-hand circularly polarized if **E** rotates clockwise to an observer viewing the oncoming light.[6] Thus the − and + signs in (2.9.8) correspond to right- and left-hand circular polarization, respectively.

In general a linearly polarized wave incident on a quarter-wave plate will be converted to an elliptically polarized wave. Linear and circular polarization are special cases of elliptical polarization.

Many isotropic media can be made birefringent by the application of an electric field. Consider, for instance, a liquid consisting of long molecules having permanent dipole moments. The existence of a permanent dipole moment implies a certain asymmetry, namely a preponderance of positive charge at one end of the molecule, and negative charge at the other. Because of collisions, the molecules are randomly oriented, and so the liquid will be macroscopically isotropic, and will not exhibit birefringence. An applied electric field, however, will tend to align the molecules, creating an anisotropy and making the liquid birefringent. This creation of birefringence by an applied electric field is called the *electro-optic effect*.

In the *Kerr electro-optic effect* the induced optic axis is parallel to the applied field, and the difference in refractive indices for light polarized parallel and perpendicular to the optic axis is proportional to the square of the applied field. Kerr cells of liquid or solid transparent media can be used together with polarizers to

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[6] This is the convention traditional in optics. It is unnatural in the sense that the right-hand rule applied to the **k** vector would suggest the opposite.

transmit or block light, depending on whether an electric field is applied. Such "light switches" have many uses in laser technology. (See Section 12.5.)

In certain crystals the induced optic axis is perpendicular to the applied electric field, and the difference in refractive indices for light polarized parallel and perpendicular to the optic axis is linearly proportional to the applied field. This electro-optic effect is called the *Pockels effect*. Pockels cells have uses similar to Kerr cells.

In certain crystals the molecules are aligned in such a way that light polarized in one direction is transmitted, whereas light polarized in the perpendicular direction is strongly absorbed. Such materials are said to be *dichroic*. (Some such materials appear colored in white light because the effect is wavelength-dependent, whence the term dichroic.) They are used, of course, as polarizers. If a wave linearly polarized in some direction $\hat{\varepsilon}$ is normally incident on a dichroic sheet in which the transmitting direction is $\hat{\mathbf{x}}$, the transmitted wave will be linearly polarized in the x direction, and will be diminished in amplitude by the factor $\hat{\varepsilon} \cdot \hat{\mathbf{x}}$. The intensity is thus diminished by $(\hat{\varepsilon} \cdot \hat{\mathbf{x}})^2 = \cos^2\theta$; this is called Malus's law. Similarly, unpolarized light incident on the polarizer will be reduced in intensity by 50%.

The most common types of polarizer are Polaroid filters, invented around 1926 by Edwin Land (the inventor of the Polaroid Land camera). One type of Polaroid filter consists of a plastic sheet in which are embedded needle-like crystals of dichroic herapathite (quinine sulfate periodide). The most common Polaroid used today is made by dipping a plastic (whose long molecules have been aligned by stretching) in iodine, which makes the plastic dichroic.

• The dichroic property of herapathite was known long before Land, but the crystals were very fragile and difficult to grow in sizes large enough to be useful. The essence of Land's idea was to embed the tiny crystals in a plastic which was stretched while soft to align them. The discovery of herapathite is rather interesting. Land writes:[7] "In the literature are a few pertinent high spots in the development of polarizers, particularly the work of William Bird Herapath, a physician in Bristol, England, whose pupil, a Mr. Phelps, had found that when he dropped iodine into the urine of a dog that had been fed quinine, little scintillating green crystals formed in the reaction liquid. Phelps went to his teacher, and Herapath then did something which I think was curious under the circumstances; he looked at the crystals under a microscope and noticed that in some places they were light where they were overlapped and in some places they were dark. He was shrewd enough to recognize that here was a remarkable phenomenon, a new polarizing material." •

We have assumed in our discussion of birefringence that the optic axis is parallel to the surface on which the field is incident. In general there will be a direction of propagation in which the refractive index is independent of the polarization; this direction defines the optic axis. (Note that the optic axis is not really a single axis, but rather refers to two opposite directions within the birefringent material.) The material may have a single optic axis, in which case it is called *uniaxial*, or it may have two optic axes, in which case it is called *biaxial*. Here we will consider only the simpler, uniaxial case.

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[7] E.H. Land, *Journal of the Optical Society of America* **41,** 957 (1951).

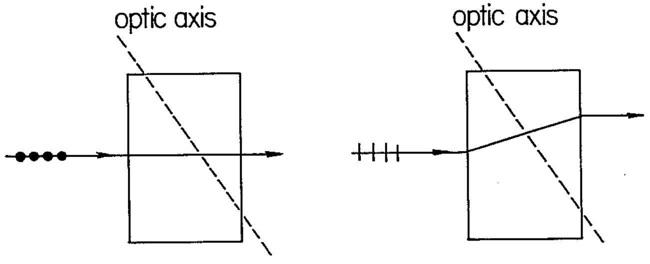
If the direction of field propagation is not parallel to the optic axis of a uniaxial crystal, there will be different refractive indices for different field polarizations. In fact only two types of wave can propagate in a uniaxial crystal, namely waves linearly polarized perpendicular to the plane formed by the optic axis and the direction of incidence, and waves linearly polarized parallel to this plane. Waves of the first type are called *ordinary*, whereas those of the second type are called *extraordinary*. Figure 2.12 illustrates what is "ordinary" and "extraordinary" about these waves. In Figure 2.12*a*, a normally incident field is linearly polarized in a direction perpendicular to the plane formed by the optic axis and the direction of incidence. In this case the field simply passes through the crystal in the expected or "ordinary" way, i.e., according to Snell's law. In Figure 2.12*b*, however, the field is linearly polarized parallel to the plane defined by the optic axis and the direction of incidence. In this case something unexpected or "extraordinary" happens: the wave is deflected at the boundaries and the rays emergent at the exit face are displaced with respect to the incident rays.

This means that if an anisotropic crystal such as Iceland spar ($CaCO_3$) is laid over a small dot on a piece of paper, we will see a double image of the dot. This phenomenon of *double refraction* (sometimes called *anomalous refraction*) was noted 400 years ago by European sailors visiting Iceland.

The refractive index for ordinary waves is denoted n_0, and is independent of the direction of propagation. The refractive index $n_e(\theta)$ for extraordinary waves, however, depends on the direction of propagation (θ) relative to the optic axis. If n_e denotes the extraordinary index for propagation normal to the optic axis (i.e., $n_e = n_e(\theta = \pi/2)$), then the following relation holds for n_0, n_e, and $n_e(\theta)$:

$$\frac{1}{n_e(\theta)^2} = \frac{\cos^2\theta}{n_0^2} + \frac{\sin^2\theta}{n_e^2} \tag{2.9.9}$$

Values of n_0 and n_e for different materials are tabulated in various handbooks.



(a) (b)

Figure 2.12 Ordinary (*a*) and extraordinary (*b*) waves in a uniaxial birefringent crystal.

If unpolarized light is incident on a doubly refracting crystal, it is separated into ordinary and extraordinary waves which are linearly polarized orthogonally to each other. This splitting of a light beam into two orthogonally polarized beams in Iceland spar (calcite) was observed by Arago and A.J. Fresnel early in the nineteenth century. Since their polarizations are orthogonal, the two beams do not interfere. This led Young (1817), and later Fresnel, to propose that light waves are transverse, for the absence of interference could not be explained if light waves were longitudinal.

2.10 SUMMARY

Equation (2.3.22) relates a property of a macroscopic medium, the refractive index, to a microscopic characteristic of the medium, the natural oscillation frequencies ω_i of bound electrons. The latter cannot be calculated with any classical model. However, this result for the refractive index is a major success of the electron oscillator model; as we have shown in Sections 2.4–2.9 it explains some important general features of optical dispersion. These features cannot be understood solely on the basis of the Maxwell equations without "atomistic" model assumptions about the medium of propagation.

The classical oscillator model of individual bound electrons was used extensively before the development of quantum mechanics. It was applied by Lorentz to the theory of nearly all the optical phenomena known at the turn of the century. Before Lorentz, physicists generally believed that electromagnetic fields were inextricably associated with continuous distributions of matter. The great contribution of Lorentz was to attribute to the electromagnetic field a truly independent existence of its own, in which, according to Einstein,

> . . .The only connection between the electromagnetic field and ponderable matter arises from the fact that elementary electric charges are rigidly attached to atomistic matter. For the latter, Newton's law of motion holds.
>
> Upon this simplified foundation Lorentz based a complete theory of all electromagnetic phenomena known at the time, including those of the electrodynamics of moving bodies. It is a work of such lucidity, consistency, and beauty as has only rarely been obtained in an empirical science.

In later chapters we will deal with concepts like "oscillator strengths" and "negative oscillators" which, though basically of a quantum-mechanical nature, reflect even by their names the conceptual framework established by Lorentz.

PROBLEMS

2.1 Write out the x component of the left and right sides of the vector identity given in (2.1.8) and show that they are equal. Remember that $(\nabla \times \mathbf{V})_x = \partial V_z/\partial y - \partial V_y/\partial z$ for any vector $\mathbf{V}$.

2.2 Verify by substitution in (2.1.14) that the z component of $\mathbf{E}$ given in (2.1.26c) is a solution of the free-space wave equation, assuming that l, m, n obey (2.1.28).

2.3 Show that the condition given in (2.1.27) is necessary if the electric field solutions given in (2.1.26) are to be "transverse," i.e., if they are to satisfy $\nabla \cdot \mathbf{E} = 0$ as assumed in (2.1.12). Note that (2.1.27) implies that only two of the amplitudes A_x, A_y, A_z can be chosen independently. This is the reason a transverse field has only two independent polarizations.

2.4 Show that the formulas for mode densities dN_ν and dN_λ, given in (2.1.30), are related through the identity $\nu = c/\lambda$.

2.5 (a) Show that the refractive index of a mixture of gases is

$$n(\omega) = \sum_i f_i n_i(\omega)$$

where $n_i(\omega)$ is the index of the ith species and f_i is its fractional concentration (number of atoms of species i divided by the total number).

(b) Using the refractive indices $n_{O_2} = 1.000272$ and $n_{N_2} = 1.000297$ of STP oxygen and nitrogen at 5890 Å, estimate the refractive index of STP air at this wavelength.

2.6 Consider a medium with $n = 1$. The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is associated with electromagnetic energy flow. Compute $\mathbf{S}$ from the solutions for $\mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ given in Table 2.2. Also compute the electromagnetic energy density $u = \frac{1}{2}(\epsilon_0 \mathbf{E}^2 + \mathbf{B}^2/\mu_0)$. By comparing $\mathbf{S}$ and u, determine the velocity of energy flow in this case.

2.7 Verify that the number of molecules per cubic centimeter in an ideal gas at standard temperature and pressure is about 2.69×10^{19}.

2.8 Assume that in the earth's ionosphere the refractive index for 100 MHz radio waves is 0.90, and that free electrons make the greatest contribution to the index.

(a) Estimate the number of electrons per cubic centimeter.

(b) Why is the contribution of positively charged ions to the refractive index much smaller?

2.9 (a) Choose an AM and an FM radio station in your area and compare their frequencies with the plasma frequency of the ionosphere.

(b) Why are automobile antennas oriented vertically rather than horizontally?

2.10* Consider the electron oscillator model for the case in which there is *no field* acting on the atom. Suppose that at $t = 0$ an electron is given the displacement $\mathbf{x}_0$ from equilibrium, and the velocity $\mathbf{v}_0$.

(a) Show that the electron coordinate $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \mathbf{x}_0 \cos \omega_0 t + \frac{\mathbf{v}_0}{\omega_0} \sin \omega_0 t$$

(b) What is the total (kinetic plus potential) energy of the electron?

(c) Using the formula (2.5.14), derive an expression for the rate at which the oscillating electron radiates electromagnetic energy. Give the rate averaged over times long compared with the period of oscillation.

(d) Show that the electron can be expected to radiate away most of its energy in a time

$$\tau = 4\pi\epsilon_0 \left(\frac{2e^2\omega_0^2}{3mc^3}\right)^{-1}$$

This is the classical picture of "spontaneous emission," which we consider in Chapter 7.

(e) Estimate numerically the "radiative lifetime" τ found in part (d) for the case of an electron oscillating at an optical frequency $\nu_0(=\omega_0/2\pi)$.

2.11 Show that the scattering cross section for radiation of frequency ω much greater than the natural oscillation frequency ω_0 is given by the Thomson formula

$$\sigma(\omega \gg \omega_0) \approx \frac{8\pi}{3} r_0^2 = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2$$

where $r_0 = e^2/4\pi\epsilon_0 mc^2$ is called the "classical electron radius." What is the magnitude of r_0?

2.12 A typical He–Ne laser operating at 6328 Å contains about five times as much He as Ne, with a total pressure of about one Torr. The length of the gain cell is about 50 cm. Estimate the fraction of laser radiation intensity lost due to Rayleigh scattering in passing a billion times through the gain cell. (Note: For STP Ne the constants in (2.4.9) are $A = 6.66 \times 10^{-5}$ and $B = 2.4 \times 10^{-11}$ cm^2.) This illustrates the fact that Rayleigh scattering is usually very weak in gas laser media.

*Starred problems are somewhat more difficult.

3 CLASSICAL THEORY OF ABSORPTION

3.1 INTRODUCTION

Most objects around us are not self-luminous but are nevertheless visible because they scatter the light that falls upon them. Most objects are *colored*, however, because they absorb light, not simply because they scatter it. The colors of an object typically arise because materials selectively absorb light of certain frequencies, while freely scattering or transmitting light of other frequencies. Thus if an object absorbs light of all visible frequencies, it is black. An object is red if it absorbs all (visible) frequencies except those our eyes perceive to be "red" (wavelengths roughly between about 6300 and 6800 Å), and so on.[1]

The physics of the absorption process is simplest in well-isolated atoms. These are found most commonly in gases. White light propagating through a gas is absorbed at the resonance frequencies of the atoms or molecules, so that one observes gaps in the wavelength distribution of the emerging light. On a spectrogram these gaps appear as bright lines on the dark, exposed background. The gaps, shown as lines in Figure 3.1, correspond to the absorption of sunlight by the atmosphere *of the sun* before the light reaches the earth. The absorbed energy is partially converted into heat (translational kinetic energy of the atoms) when excited atoms (or molecules) which have absorbed radiation collide with other particles. The absorbed radiation is also partially reradiated in all directions at the frequency of the absorbed radiation. This is called resonance radiation, or resonance fluorescence. When the pressure of the gas is increased, collisions may rapidly convert the absorbed radiation into heat before it can be reradiated. In this case the resonance radiation is said to be quenched.

Most atoms have electronic resonance frequencies in the ultraviolet, although resonances in the visible and infrared are not uncommon. Sodium, for instance, has strong absorption lines in the yellow region at 5890 and 5896 Å, the Fraunhofer "D lines," and their position is indicated in Figure 3.1.

Electronic resonances in molecules also tend to lie in the ultraviolet. We have "white" daylight because the atmosphere, consisting mostly of N_2 and O_2, does not absorb strongly at visible frequencies.

In molecules the separate atoms act approximately as if they were connected to each other by springs, so that entire atoms vibrate back and forth. Atoms are of

[1] The principal features of the electromagnetic spectrum for our purposes are summarized in Table 3 inside the cover of the book.

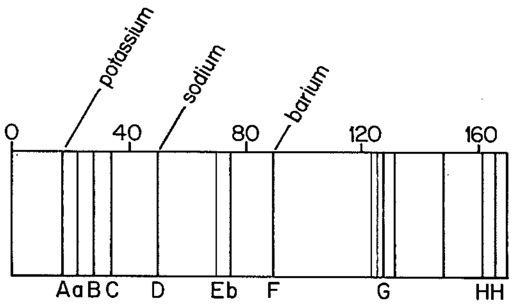


Figure 3.1 Absorption lines of the sun's atmosphere. The Fraunhofer *D* lines of sodium at 5890 and 5896 Å are not resolved in this sketch.

course much more massive (by 10^3–10^5 times) than electrons, and the natural vibrations of molecules are consequently slower. We can estimate, on the basis of this mass difference (Problem 3.1), that molecular vibration frequencies should lie in the infrared portion of the electromagnetic spectrum.

A molecule as a whole can also rotate; the resonance frequencies associated with molecular rotations lie in the microwave portion of the spectrum. Molecules therefore typically have resonances in the ultraviolet, infrared, and microwave regions of the spectrum.

Absorption in liquids and solids is much more complicated than in gases. In liquids and amorphous solids such as glass, the absorption lines have such large widths that they overlap. Water, for example, is obviously transparent in the visible, but absorbs in the near infrared, i.e., at infrared wavelengths not far removed from the visible. Its absorption curve is wide enough, in fact, that it extends into the red edge of the visible. (Figure 3.2) The weak absorption in the red portion of the visible spectrum explains why things appear green when one is sufficiently submerged under water.

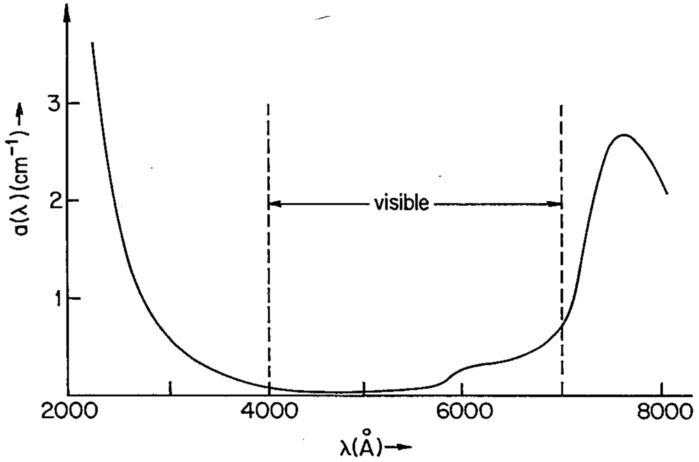


Figure 3.2 Absorption coefficient of water.

A broad absorption curve covering all visible wavelengths except those in a particular narrow band is characteristic of the molecules of a dye. The absorbed radiation is converted into heat before it can be reradiated. Such broad absorption curves and fast quenching rates require the high molecular number densities of liquids and solids.

In metals some of the atomic electrons are able to move freely about under the influence of an electromagnetic field. The fact that metals contain these "free" electrons explains, of course, why they are good conductors of electricity. In the *free-electron approximation* we may apply the dispersion formula (2.4.12). The plasma frequency ω_p for metals is usually in the ultraviolet (Problem 3.2). Thus visible frequencies ($\omega < \omega_p$) cannot penetrate into the metal. They are completely reflected, just as AM radio waves are reflected by the ionosphere. This strong reflection gives metals their shine. In a metal like gold there is also absorption, associated with the electrons that remain bound to atoms, and it is this that gives the metal a characteristic color.

In a solid that is a good electrical insulator, the electrons are tightly bound, and consequently the natural oscillation frequencies are high, typically corresponding to wavelengths less than 4000 Å. An insulator, therefore, is usually transparent in the visible but opaque in the ultraviolet. In semiconductors the natural oscillation frequencies are smaller. Silicon, for example, absorbs visible wavelengths (it is black), but transmits radiation of wavelength greater than one micron (1 micron = 1 μm).

Lattice defects (deviations from periodicity) can substantially modify the absorption spectra of crystalline solids. Ruby, for instance, is corundum (Al_2O_3) with an occasional (roughly 0.05% by weight) random substitution of Cr^{+3} ions in place of Al^{+3}. The chromium ions absorb green light and thus ruby is pink, in contrast to the transparency of pure corundum.

The variety of natural phenomena resulting from the selective absorption of certain wavelengths and the transmission of others is too broad to treat here. We mention only one important example, the "greenhouse effect."[2] Visible sunlight is transmitted by the earth's atmosphere and heats (by absorption) both land and water. The warmed earth's surface is a source of thermal radiation, the dominant emission for typical ambient temperatures being in the infrared. This infrared radiation, however, is strongly absorbed by CO_2 and H_2O vapor in the earth's atmosphere, preventing its rapid escape into space. Without this effect, the earth would be a much colder place. An increased burning of fossil fuels could conceivably enhance the greenhouse effect by increasing the level of CO_2 in the atmosphere.

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[2] The term "greenhouse effect" is actually a misnomer, originating in the observation that the glass in a greenhouse, which is transparent in the visible but opaque to the infrared, plays an absorptive role similar to that of CO_2 and H_2O in the earth's atmosphere. This effect, however, does not contribute significantly to the warming of the air inside a real greenhouse. A real greenhouse mainly prevents cooling by wind currents. This point was demonstrated experimentally by R. W. Wood (1909), although the contrary misconception persists even among scientists.

3.2 ABSORPTION AND THE LORENTZ MODEL

The strength of an electromagnetic field will be reduced in transit through a material medium if the atoms (or molecules) of the medium can absorb radiant energy. More commonly than not, in a wide variety of materials, absorption can be explained by the assumption that the Lorentz electron oscillators introduced in Chapter 2 are subject to a frictional force. The origin of a "frictional" force is itself a subject for discussion, which will be found in Section 3.9. For the moment, however, we will take a frictional force for granted, and explore its consequences.

We simply amend the Newton force law (2.2.18) to read

$$m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E}(\mathbf{R}, t) - k_s\mathbf{x} + \mathbf{F}_{\text{fric}} \tag{3.2.1}$$

and we make the simplest assumption compatible with the idea of frictional drag:

$$\mathbf{F}_{\text{fric}} = -b\mathbf{v} = -b\frac{d\mathbf{x}}{dt} \tag{3.2.2}$$

Then the Newton equation of motion (2.3.7) for an electron oscillator in a linearly polarized monochromatic plane wave takes the form

$$\frac{d^2\mathbf{x}}{dt^2} + 2\beta\frac{d\mathbf{x}}{dt} + \omega_0^2\mathbf{x} = \hat{\varepsilon}\frac{e}{m}E_0\cos(\omega t - kz) \tag{3.2.3}$$

where for later convenience we have defined

$$\beta = \frac{b}{2m}$$

As in Chapter 2 we have introduced the natural oscillation frequency

$$\omega_0 = \left(\frac{k_s}{m}\right)^{1/2} \tag{3.2.4}$$

associated with Lorentz's elastic force.

If there is no applied field, Eq. (3.2.3) becomes

$$\frac{d^2\mathbf{x}}{dt^2} + 2\beta\frac{d\mathbf{x}}{dt} + \omega_0^2\mathbf{x} = 0 \tag{3.2.5}$$

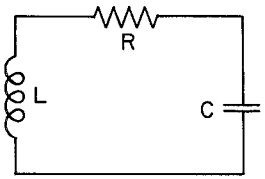

Figure 3.3 An *LRC* circuit. The charge on the capacitor obeys the equation of motion (3.2.6) for a damped oscillator.

This is the equation describing a damped oscillator. A well-known example is an *LRC* circuit (Figure 3.3), where the charge q on the capacitor satisfies the equation

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0 \tag{3.2.6}$$

In this case the natural oscillation frequency and the damping rate are determined by the fundamental parameters of the circuit:

$$\omega_0 = \left(\frac{1}{LC}\right)^{1/2} \tag{3.2.7a}$$

and

$$\beta = \frac{R}{2L} \tag{3.2.7b}$$

The solution of the differential equation (3.2.6) is

$$q(t) = (A\cos\omega_0't + B\sin\omega_0't)\,e^{-\beta t} \tag{3.2.8a}$$

where

$$\omega_0' = (\omega_0^2 - \beta^2)^{1/2} \tag{3.2.8b}$$

Under most conditions of interest the oscillator will be significantly underdamped [see Eq. (3.3.10)] and we can replace ω_0' by ω_0. Since (3.2.6) is a second-order linear differential equation, its solution has two constants of integration which are determined by the initial conditions for $q(t)$ and $dq(t)/dt$. We have denoted these two constants by A and B.

If the LRC circuit is driven by a sinusoidal emf (Figure 3.4),

$$V(t) = V_0\cos(\omega t - \theta) \tag{3.2.9}$$

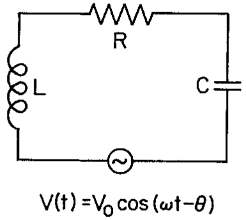

Figure 3.4 An *LRC* circuit with a sinusoidal emf. The charge on the capacitor obeys the equation of motion (3.2.10) for a sinusoidally driven, damped oscillator.

then q satisfies the *forced-oscillator* equation

$$\frac{d^2q}{dt^2} + 2\beta\frac{dq}{dt} + \omega_0^2 q = \frac{V_0}{L}\cos(\omega t - \theta) \tag{3.2.10}$$

where ω_0 and β are given by (3.2.7). This is just a scalar version of the electron oscillator vector equation (3.2.3), with L corresponding to the electron's mass/charge ratio

$$L = \frac{m}{e} \tag{3.2.11}$$

and θ corresponding to the field phase at the position of the atom:

$$\theta = kz = \frac{2\pi z}{\lambda} \tag{3.2.12}$$

In contrast to the homogeneous solution (3.2.8a), which decays to zero, the solution to the forced-oscillator equation (3.2.10) is a steady sinusoidal oscillation with an amplitude depending on ω and ω_0. The amplitude has a maximum when $\omega \approx \omega_0$, and one says that the circuit of Figure 3.4 exhibits a resonance. From (3.2.7a) we see that this resonance condition is met when the capacitance is

$$C = \frac{1}{\omega^2 L} \tag{3.2.13}$$

When the resonance condition is approached by tuning the capacitance to the resonance value (3.2.13), the amplitude of the oscillating current in the circuit increases dramatically, as shown in Figure 3.5. This resonant enhancement is used in simple radio receivers, where a variable capacitor permits tuning to various broadcast frequencies.

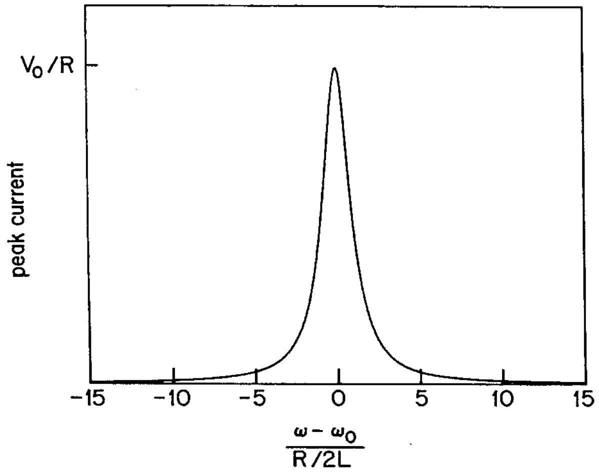

Figure 3.5 The amplitude of the oscillating current in an *LRC* circuit with emf (3.2.9). The current oscillation is at the driving frequency ω, and has maximum amplitude when the resonance condition $\omega = \omega_0 = (1/LC)^{1/2}$ is satisfied.

The interaction of an atom with a monochromatic field is similarly enhanced when

$$\omega = \omega_0 \tag{3.2.14}$$

i.e., when the frequency of the field coincides with a natural oscillation frequency of a bound electron. This enhancement of the interaction is already implied by our result (2.3.14b) for the refractive index. However, that result is obviously undefined if $\omega = \omega_0$. A frictional force in the electron oscillator model allows us to understand formulas like (2.3.14b) even for $\omega = \omega_0$, while also providing the physical mechanism for the absorption of electromagnetic energy.

3.3 COMPLEX POLARIZABILITY AND INDEX OF REFRACTION

The equation (3.2.3) for the electron oscillator with damping is most easily solved by first writing it in complex form:

$$\frac{d^2\mathbf{x}}{dt^2} + 2\beta\frac{d\mathbf{x}}{dt} + \omega_0^2\mathbf{x} = \hat{\boldsymbol{\varepsilon}}\frac{e}{m}E_0 e^{-i(\omega t - kz)} \tag{3.3.1}$$

where we follow the convention of writing $E_0 \cos(\omega t - kz)$ as $E_0\, e^{-i(\omega t - kz)}$. This means that $\mathbf{x}(t)$ in Eq. (3.3.1) is also regarded mathematically as a complex quantity in our calculations, but only its real part is physically meaningful. In other words, we may defer the process of taking the real part of (3.3.1) until *after* our calculations, at which point the real part of our solution for $\mathbf{x}(t)$ is the (real) electron displacement. This approach is standard in solving linear equations, but there are pitfalls that can arise in nonlinear problems. [See Chapter 17, where modifications of Eq. (3.3.1) are used in an introduction to nonlinear optics.]

We solve (3.3.1) by temporarily writing

$$\mathbf{x}(t) = \mathbf{a}\, e^{-i(\omega t - kz)} \tag{3.3.2}$$

and after inserting this in (3.3.1) we obtain

$$(-\omega^2 - 2i\beta\omega + \omega_0^2)\,\mathbf{a} = \hat{\boldsymbol{\varepsilon}}\,\frac{e}{m}\,E_0 \tag{3.3.3}$$

Therefore the assumed solution (3.3.2) satisfies Eq. (3.3.1) if

$$\mathbf{a} = \frac{-\hat{\boldsymbol{\varepsilon}}(e/m)\,E_0}{\omega^2 - \omega_0^2 + 2i\beta\omega} \tag{3.3.4}$$

and the physically relevant solution is therefore

$$\mathbf{x}(t) = \mathrm{Re}\left(\frac{\hat{\boldsymbol{\varepsilon}}(e/m)\,E_0\, e^{-i(\omega t - kz)}}{\omega_0^2 - \omega^2 - 2i\beta\omega}\right) \tag{3.3.5}$$

Note that (3.3.5) actually gives only the steady-state solution of (3.3.1). Any solution of the homogeneous version of (3.3.1) can be added to (3.3.5), and the total will still be a solution of (3.3.1). The homogeneous version is

$$\frac{d^2\mathbf{x}_{\text{hom}}}{dt^2} + 2\beta\,\frac{d\mathbf{x}_{\text{hom}}}{dt} + \omega_0^2\mathbf{x}_{\text{hom}} = 0 \tag{3.3.6}$$

and its general solution is an obvious vectorial extension of (3.2.8a):

$$\mathbf{x}_{\text{hom}} = [\mathbf{A}\cos\omega_0' t + \mathbf{B}\sin\omega_0' t]\, e^{-\beta t} \tag{3.3.7}$$

where again

$$\omega_0' = (\omega_0^2 - \beta^2)^{1/2} \approx \omega_0 \tag{3.3.8}$$

We will usually neglect the homogeneous part of the full solution to (3.3.1). This is obviously an approximation. The approximation is however an excellent one whenever

$$t \gg 1/\beta \tag{3.3.9}$$

ponent (3.3.7) because it makes only a short-lived transient contribution to the solution.

Even though the homogeneous damping time, or lifetime $\tau_0 = 1/\beta$, is very short, it is not the shortest time in the problem. Typically the oscillation periods $T = 2\pi/\omega$ and $T_0 = 2\pi/\omega_0$ associated with the natural oscillation frequency ω_0 or the forcing frequency ω are very much shorter. In the case of ordinary optically transparent materials such as atomic vapors, glasses, and many crystals and liquids, both ω_0 and ω are typically in the neighborhood of 10^{15} sec^{-1}, and β falls in a wide range of much smaller frequencies:

$$\beta \approx 10^6\text{–}10^{12}\ \text{sec}^{-1} \ll \omega_0,\ \omega \tag{3.3.10}$$

Relations (3.3.9) and (3.3.10), taken together, imply that times of physical interest must be much longer than an optical period:

$$t \gg \beta^{-1} \gg \omega_0^{-1},\ \omega^{-1} \tag{3.3.11}$$

That is, steady-state solutions of (3.3.1) are valid for times that are many periods of oscillator vibration ($T_0 = 2\pi/\omega_0$) and forced vibration ($T = 2\pi/\omega$) removed from $t = 0$, but they cannot be used to predict the oscillator's response within the first few cycles after $t = 0$. This is, however, a restriction of no real significance in optical physics, as it is equivalent to

$$t \gg 10^{-15}\ \text{sec}\ (=10^{-3}\ \text{ps}) \tag{3.3.12}$$

This is a time span one or two orders of magnitude smaller than can presently be resolved optically.

The steady-state solution (3.3.5) is very close to the solution (2.3.8) for the undamped oscillator. It implies that the electric field induces in an atom a dipole moment $\mathbf{p} = e\mathbf{x}$, or $\mathbf{p} = \Sigma_j\, e\mathbf{x}_j$ in the case of many electrons:

$$\mathbf{p} = \mathrm{Re}\left(\hat{\boldsymbol{\varepsilon}}\,\frac{e^2}{m}\,\frac{E_0\, e^{-i(\omega t - kz)}}{\omega_0^2 - \omega^2 - 2i\beta\omega}\right) \tag{3.3.13}$$

or

$$\mathbf{p} = \mathrm{Re}\left(\hat{\boldsymbol{\varepsilon}}\frac{e^2}{m}E_0\, e^{-i(\omega t - kz)} \sum_{j=1}^{Z} \frac{1}{\omega_j^2 - \omega^2 - 2i\beta_j\omega}\right)$$

The real part can be found explicitly to be

$$\mathbf{p} = \hat{\boldsymbol{\varepsilon}}\,\frac{e^2}{m}\left(\frac{(\omega_0^2 - \omega^2)\,E_0\cos(\omega t - kz) + 2\beta\omega E_0\sin(\omega t - kz)}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\right) \tag{3.3.14}$$

Because of the frictional damping (i.e., because $\beta \neq 0$) the dipole moment no longer oscillates completely in phase with the electric field as it did in (2.3.8). The new term proportional to $\sin(\omega t - kz)$ signifies the existence of a phase lag in the dipole response. Thus there is no single real polarizability coefficient that can be identified as the ratio of the dipole moment and the electric field strength.

It is possible nevertheless, and generally very convenient, to introduce a complex polarizability. This is done by recognizing that (3.3.2) can be used to define a complex dipole moment **p**:

$$\mathbf{p} = e\mathbf{x} = e\mathbf{a}\, e^{-i(\omega t - kz)} \tag{3.3.15}$$

The complex polarizability α is defined by the relation between complex moment and complex field:

$$\mathbf{p} = \alpha(\omega)\, \hat{\varepsilon} E_0\, e^{-i(\omega t - kz)} \tag{3.3.16}$$

In the present case, by comparing (3.3.13) and (3.3.16) we easily identify the complex polarizability of a Lorentzian atom to be

$$\begin{aligned} \alpha(\omega) &= \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} \\ &= \frac{e^2}{m} \frac{\omega_0^2 - \omega^2 + 2i\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \end{aligned} \tag{3.3.17}$$

or in the case of many electrons,

$$\begin{aligned} \alpha(\omega) &= \sum_{j=1}^{Z} \frac{e^2/m}{\omega_j^2 - \omega^2 - 2i\beta_j\omega} \\ &= \sum_{j=1}^{Z} \frac{e^2}{m} \frac{\omega_j^2 - \omega^2 + 2i\beta_j\omega}{(\omega_j^2 - \omega^2)^2 + 4\beta_j^2\omega^2} \end{aligned} \tag{3.3.18}$$

Given the complex polarizability (3.3.17) or (3.3.18), the complex polarization density is

$$\mathbf{P} = N\mathbf{p} = N\alpha(\omega)\, \hat{\varepsilon} E_0\, e^{-i(\omega t - kz)} \tag{3.3.19}$$

Using this polarization density in the wave equation (2.1.13), together with the complex form of the assumed solution (2.3.1), we obtain

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) \hat{\varepsilon}\, E_0\, e^{-i(\omega t - kz)}$$

$$\omega^2 N\alpha(\omega) \quad \hat{\varepsilon} E \quad e^{-i(\omega t - kz)} \qquad (3.3.20)$$

Therefore k must satisfy the dispersion relation

$$\begin{aligned} k^2 &= \frac{\omega^2}{c^2}\left(1 + \frac{N\alpha(\omega)}{\epsilon_0}\right) \\ &= \frac{\omega^2}{c^2} n^2(\omega), \end{aligned} \tag{3.3.21}$$

just as in Eq. (2.3.13).

In this case, because $\alpha(\omega)$ is complex the refractive index is also a complex number:

$$\begin{aligned} n^2(\omega) &= 1 + \frac{Ne^2/m\epsilon_0}{\omega_0^2 - \omega^2 - 2i\beta\omega} \\ &= 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0^2 - \omega^2 + 2i\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \\ &= [n_R(\omega) + in_I(\omega)]^2 \end{aligned} \tag{3.3.22}$$

The most important consequence of these results is that the electric field in the medium behaves differently from the field discussed in Chapter 2 because $n(\omega)$ is now complex:

$$\begin{aligned} \mathbf{E}(z, t) &= \hat{\varepsilon} E_0\, e^{-i(\omega t - kz)} \\ &= \hat{\varepsilon} E_0\, e^{-i\omega[t - n(\omega) z/c]} \\ &= \hat{\varepsilon} E_0\, e^{-[n_I(\omega)]\omega z/c}\, e^{-i\omega\{t - [n_R(\omega)] z/c\}} \end{aligned} \tag{3.3.23}$$

Note that $\mathbf{E}(z, t)$ is no longer purely oscillatory. Due to $n_I(\omega)$, the field decays with increasing distance of propagation. Since the intensity is proportional to the square of the (real) electric field [recall Eq. (2.6.4) and (2.6.8)], the intensity shows exponential decay with z:

$$I_\omega(z) = I_\omega(0)\, \left(e^{-[n_I(\omega)]\omega z/c}\right)^2 = I_0\, e^{-a(\omega) z} \tag{3.3.24}$$

where we call $a(\omega)$ the *absorption coefficient* or *extinction coefficient*:

$$\begin{aligned} a(\omega) &= 2[n_I(\omega)]\, \omega/c \\ &= \frac{2Ne^2}{\epsilon_0 mc} \sum_j \frac{\beta_j\omega^2}{(\omega_j^2 - \omega^2)^2 + 4\beta_j^2\omega^2} \end{aligned} \tag{3.3.25}$$

As in (2.3.23) we have used $n \approx 1$. This is a very important result, and we will return to it shortly.

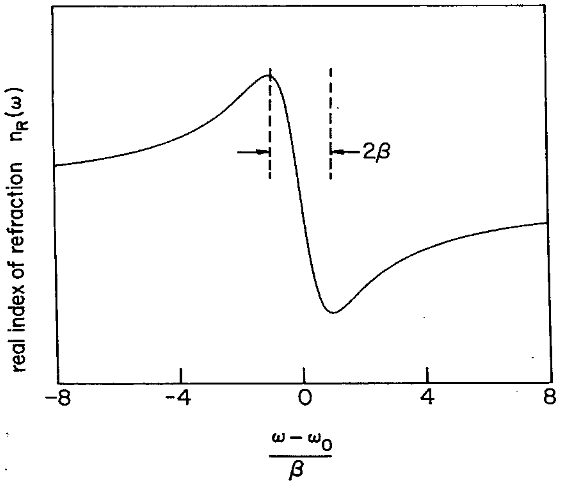

Figure 3.6 Anomalous dispersion curve for a collision-broadened absorption line.

plex refractive index is therefore what would ordinarily be called the "refractive index." This refractive index is plotted versus frequency in Figure 3.6. On the low-frequency side of each resonance frequency, $n_R(\omega)$ increases with increasing frequency, i.e., we have "normal dispersion" (Section 2.4). However, when ω gets within β_j of ω_j, $n_R(\omega)$ begins *decreasing* with increasing frequency. This decrease continues until ω is more than β_j from ω_j on the high-frequency side, whereupon it again increases with increasing frequency. Because most media show normal dispersion at optical frequencies, the negative slope of the dispersion curve near an absorption line was historically termed *anomalous dispersion.*

• Anomalous dispersion was observed by R. W. Wood in 1904. Wood studied the dispersion of light at frequencies near the sodium D lines (5890 and 5896 Å). The basic idea of Wood's experiment is sketched in Figure 3.7. Light enters a tube in which sodium vapor is produced by heating sodium. The vapor pressure decreases upwards in the tube, so that for normal dispersion the light would be bent downward, in the direction of greater density and refractive index. The vapor thus acts as a kind of prism. The light emerging at the other end of the tube is focused onto the entrance slit of a spectroscope. Wood writes:

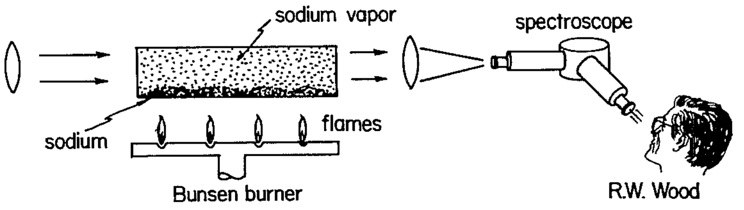

Figure 3.7 One of R. W. Wood's experiments on anomalous dispersion in sodium vapor.

> On heating the tube, the sodium prism deviates the rays of different wave-length up or down by different amounts, curving the spectrum into two oppositely directed branches. The spectrum on the green side of the D lines will be found to bend down in the spectroscope, which means that the rays are deviated upwards in passing through the sodium tube, since the spectroscope inverts the image of its slit. This means that this phase velocity is greater in the sodium vapor than in vacuo, or the prism acts for these rays like an air prism immersed in water. The red and orange region is deviated in the opposite direction; these rays are therefore retarded by the vapor.

In other words, the refractive index on the low-frequency side of resonance was observed to be greater than unity, whereas on the high-frequency side it was less than unity. This is the behavior shown in Figure 3.6. In fact Wood's measured curve of refractive index versus frequency showed exactly the "anomalous dispersion" form predicted by the electron oscillator model. •

3.4 POLARIZABILITY AND INDEX OF REFRACTION NEAR A RESONANCE

Most of the time we will be primarily interested in the response of the dipoles that are very nearly resonant with an applied field. These dipoles will usually be a small minority of the dipoles present. The sharpness of their resonant response (recall Figure 3.5) makes them particularly important. However, the other dipoles in the far off-resonant "background" can be so numerous that they also make a significant contribution to the polarizability and index of refraction, and we cannot overlook them.

Equation (3.3.18) shows that the polarizability is additive over all dipole response frequencies. Thus we can write

$$\alpha(\omega) = \alpha_b(\omega) + \alpha_r(\omega) \tag{3.4.1}$$

where α_b and α_r are the contributions from "background" and "resonant" dipoles, respectively. The background dipoles may reside in an actual host material, in which the atoms with the resonant dipoles are embedded, or they may be dipoles associated with nonresonant oscillations in the same atoms as the resonant dipoles. In either event, the relations (3.4.1) and (3.3.21) imply

$$n^2(\omega) = 1 + \sum_i \frac{N_{bi}}{\epsilon_0}\alpha_{bi}(\omega) + \frac{N_r}{\epsilon_0}\alpha_r(\omega) \tag{3.4.2}$$

where we have indicated a sum over all background species.

The first two terms in (3.4.2) determine $n_b(\omega)$, the index of refraction of the background or host material. Thus we will write

$$\begin{aligned} n^2(\omega) &= n_b^2(\omega) + \frac{N_r\alpha_r(\omega)}{\epsilon_0} \\ &= n_b^2(\omega)\left(1 + \frac{N_r\alpha_r(\omega)}{n_b^2(\omega)\,\epsilon_0}\right) \\ &= n_b^2(\omega)\left(1 + \frac{N_r\alpha_r(\omega)}{\epsilon_b(\omega)}\right) \end{aligned} \tag{3.4.3}$$

where $\epsilon_b = n_b^2\epsilon_0$ is the dielectric permittivity of the background. If the resonant atoms are present in a monatomic beam, then the background material is vacuum or nearly so and the background contributions can largely be ignored. Even in an atomic vapor n_b can be taken to be unity to three or four significant figures. However, in laser physics, the background material is frequently a solid or liquid. For example, the ruby laser operates because of dipoles associated with chromium ions thinly dispersed throughout a solid lattice (the crystal called corundum), and the dye molecules of a dye laser are dissolved in a liquid solvent (for example ethanol). Then n_b is significantly different from unity, typically in the range 1.3–2.0. We will write n_b in place of $n_b(\omega)$ hereafter because the resonances of the background are typically in the infrared or ultraviolet and n_b is effectively constant at optical frequencies.

The resonant dipoles do not make a correspondingly large contribution, since they are usually present in such small concentrations. The concentration of the chromium ions in ruby, for example, may be only 10^{19} per cm^3 or even less, much smaller than typical solid densities. As a consequence the last term in (3.4.3) is typically much smaller than unity. Then the total index of refraction can be expressed compactly as follows:

$$\begin{aligned} n(\omega) &= n_b\left(1 + \frac{N_r\alpha_r(\omega)}{\epsilon_b}\right)^{1/2} \\ &\approx n_b + \frac{N_r\alpha_r(\omega)}{2n_b\epsilon_0} \end{aligned} \tag{3.4.4}$$

where we have again used $\epsilon_b = n_b^2\epsilon_0$ after expanding the square root and keeping only the first term in the binomial series $(1 + x)^{1/2} = 1 + x/2 + x^2/8 + \cdots$.

Now we must consider what we mean by "near to resonance." Note in (3.3.17) that when $\beta = 0$ the imaginary part of $\alpha(\omega)$ vanishes and the real part reduces to (2.3.10). In any event, if ω is far enough from the resonance frequencies ω_j, we can put $\beta = 0$ without affecting the result appreciably. It should be clear then that "far from resonance" is only a relative term, relative to the damping coefficient β. For any resonance frequency ω_j, then, "far from resonance" means

$$|\omega_j - \omega| >> \beta_j \tag{3.4.5a}$$

and "near to resonance" means

$$|\omega_j - \omega| \leq \beta_j \tag{3.4.5b}$$

A significant contribution to $\alpha(\omega)$ can come from a resonance if the associated β is small enough. Suppose there is one frequency $\omega_j = \omega_0$ close enough to ω to satisfy (3.4.5b) and all others satisfy the off-resonance condition (3.4.5a). For clarity we will label the resonant damping coefficient β without a subscript. Then we can write

$$\alpha_r(\omega) = \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 - 2i\beta\omega}$$

The resonant part of $\alpha(\omega)$ can be written in a still simpler form if ω is close enough to ω_0 to justify the approximation

$$|\omega_0 - \omega| << \omega, \omega_0 \tag{3.4.6}$$

which is always guaranteed in practice whenever the earlier approximation $|\omega_0 - \omega| \leq \beta$ is valid. In this case we can write

$$\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega(\omega_0 - \omega), \tag{3.4.7}$$

and under this condition we have

$$\alpha_r(\omega) = \frac{e^2/2m\omega}{\omega_0 - \omega - i\beta}. \tag{3.4.8}$$

When the field frequency ω is far removed from *all* the resonance frequencies ω_j of the medium, the complex polarizability (3.3.17) reduces to the real polarizability (2.3.10). In this case the refractive index predicted by the electron oscillator model has been discussed in Chapter 2. For frequencies ω near to any of the ω_j, however, the friction coefficient β becomes important. For example, it is just because β is not zero that the refractive index does not become infinite whenever $\omega = \omega_j$, as is (erroneously) predicted by (2.3.14).

The real and imaginary parts of the index of refraction can now be identified easily, using (3.4.8) for $\alpha_r(\omega)$, and we find

$$n_R(\omega) = n_{bR} + \frac{Ne^2}{4n_{bR}\epsilon_0 m\omega}\frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + \beta^2} \tag{3.4.9}$$

$$n_I(\omega) = n_{bI} + \frac{Ne^2}{4n_{bR}\epsilon_0 m\omega}\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \tag{3.4.10}$$

Here we have written n_{bR} and n_{bI} for the real and imaginary parts of n_b. Also we have assumed $n_{bR} >> n_{bI}$. Finally, by comparison with (3.3.25) and (3.4.10), we obtain the absorption coefficient due to the resonance frequency ω_0:

$$\begin{aligned} a(\omega) &= 2n_I(\omega)\,\omega/c \\ &= a_b(\omega) + \frac{Ne^2}{2n_{bR}\epsilon_0 mc}\,\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \end{aligned} \tag{3.4.11}$$

where $a_b(\omega) = 2n_{bI}\,\omega/c$ is the background absorption coefficient.

3.5 LORENTZIAN ATOMS AND RADIATION IN CAVITIES

The Newton–Lorentz equation for the response of an atomic dipole to an applied radiation field was given in (3.2.3) under the assumption that the radiation took the form of a traveling wave. That is, in complex notation, the electric field was assumed to have the form

$$\mathbf{E}(z, t) = \hat{\boldsymbol{\varepsilon}} E_0\, e^{-i(\omega t - kz)} \tag{3.5.1}$$

This is not appropriate for dipoles in cavities, where the electric field takes the form of a standing wave:

$$\mathbf{E}(z, t) = \hat{\boldsymbol{\varepsilon}} E_n \sin k_n z\, e^{-i\omega t} \tag{3.5.2}$$

where

$$k = k_n = n\pi/L, \qquad n = 1, 2, 3, \ldots \tag{3.5.3}$$

as we indicated in Eq. (1.3.2) and derived in Section 2.1.

In this section we will examine the polarizability of atoms exposed to a standing-wave field, and the radiation emitted by these atoms into the cavity. The principal consequences are a new expression for the relation between k and ω and the discovery that a classical laser of Lorentz dipoles can't work.

In free space [recall (2.3.13)] we specified ω and used the coupled Maxwell–Newton equations to find $k = k(\omega)$, and this dispersion relation defined the index of refraction: $n(\omega) = k(\omega)\,c/\omega$, as in (3.3.21). In a cavity, we specify the cavity length L which first determines the wave vector $k = k_n = n\pi/L$ [recall (1.3.2)], but not the frequency ω. We will use the coupled Maxwell–Newton equations to find $\omega = \omega(k_n) \neq \omega_n$. That is, we will find that the presence of dipoles in the cavity will bias ω, the actual oscillation frequency of the field, away from the natural frequency of the cavity mode, $\omega_n = n\pi c/L$.

First we rewrite (3.2.3) using (3.5.2) and obtain

$$\frac{d^2\mathbf{x}}{dt^2} + 2\beta\frac{d\mathbf{x}}{dt} + \omega_0^2\mathbf{x} = \hat{\boldsymbol{\varepsilon}}\frac{e}{m}E_n \sin k_n z\, e^{-i\omega t} \tag{3.5.4}$$

The solution of this equation is the obvious analog of (3.3.2):

$$\mathbf{x} = \mathbf{a} \sin k_n z\, e^{-i\omega t} \tag{3.5.5}$$

where the amplitude **a** can be found easily by substitution into (3.5.4). It satisfies (3.3.3) exactly. In other words the atomic polarizability $\alpha(\omega)$ remains as derived in (3.3.17), even though the atoms are in a standing wave.

Next we determine the field amplitude. The appropriate Maxwell wave equation for a cavity is the same as (2.1.13), except that cavity losses can be included by adding an ohmic current $\mathbf{J} = \sigma\mathbf{E}$ to the right side of (2.1.4). Then we obtain

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\sigma}{\epsilon_0 c^2}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(z, t) = \frac{1}{\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}(z, t) \tag{3.5.6}$$

Here the second term represents the effect of ohmic losses, such as would be due to a finite conductivity σ (Problem 3.3). This is a common method for modeling cavity losses in laser theory.

The polarization is defined to be $\mathbf{P} = Ne\mathbf{x}$, as before, so we can use (3.5.5) to evaluate the derivatives on the right side of (3.5.6) and use (3.5.2) for computing the derivatives on the left side. After differentiating we can cancel the common factor $\hat{\boldsymbol{\varepsilon}} E_n \sin k_n z\, e^{-i\omega t}$ on both sides to get:

$$-k_n^2 + i\left(\frac{\omega\sigma}{\epsilon_0 c^2}\right) + \left(\frac{\omega}{c}\right)^2 = -\left(\frac{\omega}{c}\right)^2 \frac{Ne^2/\epsilon_0 m}{\omega_0^2 - \omega^2 - 2i\beta\omega}$$

Now we use $k_n = \omega_n/c$, and the near-resonance approximation (3.4.7) twice:

$$\omega_0^2 - \omega^2 \approx 2\omega(\omega_0 - \omega) \tag{3.5.7a}$$

$$\omega^2 - \omega_n^2 \approx 2\omega(\omega - \omega_n) \tag{3.5.7b}$$

to get

$$\begin{aligned} \omega - \omega_n + i\frac{\sigma}{2\epsilon_0} &= -\frac{Ne^2}{4\epsilon_0 m}\,\frac{\omega_0 - \omega + i\beta}{(\omega_0 - \omega)^2 + \beta^2} \\ &= \tfrac{1}{2}(\delta + ig)\,c \end{aligned} \tag{3.5.8}$$

where we have defined g and δ as abbreviations for:

$$g = -\frac{Ne^2}{2\epsilon_0 mc}\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \tag{3.5.9}$$

and

$$\begin{aligned}\delta &= -\frac{Ne^2}{2\epsilon_0 mc}\frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + \beta^2}\\ &= \frac{g(\omega_0 - \omega)}{\beta}\end{aligned} \tag{3.5.10}$$

We note immediately that all reference to the field amplitude has dropped out in the step from (3.5.6) to the solution (3.5.8), as it did in the similar step between Eqs. (3.3.20) and (3.3.21). What remains is the consistency condition (3.5.8) on the parameters of the interaction. That is, (3.5.8) is the dispersion relation for the cavity.

Let us now solve for g and δ. By matching imaginary parts of (3.5.8) we quickly determine

$$g = \sigma/\epsilon_0 c \tag{3.5.11}$$

Next we look at the real parts of (3.5.8). With the aid of (3.5.10) we find the simple relation

$$\omega_n - \omega = \frac{gc}{2\beta}(\omega - \omega_0) \tag{3.5.12}$$

We can interpret this second relation as a condition on the oscillation frequency ω. Note that if ω is below the cavity frequency ω_n, the left side of (3.5.12) is positive and the right side shows that ω must then lie above the atomic dipole frequency ω_0. Conversely, if ω is above ω_n, then it must be below ω_0. In other words, no matter whether $\omega_0 > \omega_n$ or $\omega_n > \omega_0$, the operating frequency lies between the cavity frequency and the dipole frequency. This is called *frequency pulling*; the interaction with the atomic dipoles pulls the electric field frequency away from the free-space cavity frequency and toward the dipole frequency. An explicit solution of (3.5.12) is

$$\begin{aligned}\omega &= \frac{\beta\omega_n + (gc/2)\,\omega_0}{\beta + gc/2}\\ &\approx \omega_n + \frac{gc}{2\beta}(\omega_0 - \omega_n) \qquad (\beta >> gc/2)\end{aligned} \tag{3.5.13}$$

It is possible to give a physical interpretation to the equation for g as well. If we were to allow E_n in Eq. (3.5.2) to be time-dependent, then upon substitution into Maxwell's wave equation (3.5.6) we would obtain a differential equation for $E_n(t)$ instead of the consistency equation (3.5.11). We would find that $E_n(t)$ grows exponentially in time if $g > \sigma/\epsilon_0 c$. Thus gc is the classical *gain coefficient* for the interaction of radiation with atomic dipoles in a cavity (Problem 3.4).

We could go on and formulate immediately a classical theory of laser action. For example, the equality in (3.5.11) gives the value of $g = \sigma/\epsilon_0 c$ at which amplification is first possible. This is the *threshold gain*, usually denoted g_t. Unfortunately, none of this is realistic because (3.5.11) cannot be satisfied. That is, from (3.5.9) we see immediately that g is intrinsically negative. Radiation in the cavity will only be damped and never amplified by classical dipoles. A classical laser theory based on the linear electron oscillator model is not possible.

The negative value of g is inherent in the classical theory. It requires a quantum-mechanical treatment of the light–matter interaction to understand how g can be made positive. Apart from this detail, it is remarkable how much of the present classical formulation survives the transition to quantum theory. For example, except for its sign, the form of the gain coefficient will turn out to be exactly correct. The frequency-pulling equation (3.5.12) is exactly correct as it stands. The threshold condition (3.5.11) is correct. We will find how to make g positive in Chapter 7, and in so doing will find other missing elements of laser theory, such as saturation and power broadening.

3.6 THE ABSORPTION COEFFICIENT

We can associate the energy absorbed from an electromagnetic wave by an atom with the work done by the wave on the Lorentzian oscillators. In classical mechanics the rate at which work is done on an atom when a force **F** is exerted on it is $dW_A/dt = \mathbf{F}\cdot\mathbf{v}$. In the electron oscillator model the force exerted on an electron by the monochromatic field (2.3.1) is simply the Lorentz force appearing on the right side of (2.2.18):

$$\mathbf{F}_{\text{em}} = e\boldsymbol{\mathcal{E}}E_0\cos(\omega t - kz) \tag{3.6.1}$$

in which case we can write

$$\frac{dW_A}{dt} = \mathbf{E}\cdot\frac{d\mathbf{p}}{dt} \tag{3.6.2}$$

This expression does not lead to energy absorption by the oscillator if **p** is in phase with **E**. In this section we focus attention on an oscillator near to resonance, for which $\alpha(\omega)$ has a significant imaginary (quadrature) part and for which energy absorption does occur. We can use (3.4.8) to obtain

$$\mathbf{p} = \hat{\varepsilon}\frac{e^2}{2m\omega}\frac{1}{\omega_0 - \omega - i\beta}E_0 e^{-i(\omega t - kz)} \tag{3.6.3a}$$

which corresponds to the (real) physical moment:

$$\mathbf{p}(t) = \hat{\varepsilon}e[U\cos(\omega t - kz) - V\sin(\omega t - kz)] \tag{3.6.3b}$$

The coefficients U and V are easily found by computing the real part of (3.6.3a) and comparing with (3.6.3b):

$$U = +\frac{eE_0}{2m\omega}\frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + \beta^2} \tag{3.6.4}$$

and

$$V = -\frac{eE_0}{2m\omega}\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \tag{3.6.5}$$

The corresponding solution obtained without damping would have no quadrature component corresponding to V. The existence of the quadrature component is critical to our discussion of absorption, as we now demonstrate. From Eq. (3.6.3b) we obtain

$$\frac{d\mathbf{p}}{dt} = -\omega\hat{\varepsilon}e[U\sin(\omega t - kz) + V\cos(\omega t - kz)] \tag{3.6.6}$$

Therefore the rate at which the dipole energy changes is given by

$$\begin{aligned}\frac{dW_A}{dt} &= -eE_0[\omega U\sin(\omega t - kz)\cos(\omega t - kz) + \omega V\cos^2(\omega t - kz)] \\ &= e\omega E_0[-\tfrac{1}{2}U\sin(2\omega t - 2kz) - V\cos^2(\omega t - kz)]\end{aligned} \tag{3.6.7}$$

Notice that the dipole's energy gain has two distinct contributions. The first term oscillates extremely rapidly and is zero on average, and thus does not give rise to any permanent change in energy. The second term, however, is always positive-definite and corresponds to a steady decrease in field energy with time. Then the rate of change of electromagnetic field energy, equal and opposite to dW_A/dt, is effectively governed by the second term alone:

$$\begin{aligned}\frac{dW_{\text{em}}}{dt} &= e\omega E_0 V\cos^2(\omega t - kz) \\ &= -\frac{e^2}{2m}\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2}E_0^2\cos^2(\omega t - kz)\end{aligned} \tag{3.6.8}$$

where we have used the expression (3.6.5) for V.

Thus we may express the rate (3.6.8) at which electromagnetic energy is absorbed by an atom in terms of the magnitude of the Poynting vector at the atom [recall (2.6.4)]:

$$\frac{dW_{\text{em}}}{dt} = -\frac{e^2}{2\epsilon_0 mc}\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2}|\mathbf{S}| \tag{3.6.9}$$

This result is similar to (2.6.5). Both equations show that dW_{em}/dt is proportional to $|\mathbf{S}|$. Of course (2.6.5) gives the rate of change of electromagnetic energy in a light beam due to scattering, whereas (3.6.9) gives the rate due to absorption.

The similarity of (3.6.9) to (2.6.5) means that we may define an *absorption cross section*:

$$\sigma(\omega) = \frac{e^2}{2\epsilon_0 mc}\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \tag{3.6.10}$$

We may follow the same steps, leading from (2.6.6) to the extinction coefficient (2.6.16) due to scattering, to obtain the extinction coefficient due to absorption in a medium of N atoms per unit volume:

$$a(\omega) = N\sigma(\omega) = \frac{Ne^2}{2\epsilon_0 mc}\frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \tag{3.6.11}$$

This extinction coefficient is usually called simply the *absorption coefficient*. The intensity of the incident wave after propagating a distance z into the absorbing medium is

$$I_\omega(z) = I_\omega(0)\,e^{-a(\omega)z} \tag{3.6.12}$$

just as in the case (2.6.15) when the incident wave is attenuated because of scattering.

Equation (3.6.12) is identical to (3.3.24). We have simply obtained the same physical result for the field attenuation due to absorption using two approaches. In the first approach, leading to (3.3.24), absorption was associated with the imaginery part of the complex refractive index. In this section we have obtained the same result via the rate at which a single atom absorbs energy from the field. The two approaches are equivalent. Keep in mind, however, that the absorption coefficient derived here is physically distinct from the extinction coefficient due to scattering derived in Section 2.6. Both lead to exponential attenuation of intensity, and the total extinction coefficient includes both.

The absorption coefficient is often written in terms of the circular frequency ν,

$$\nu = \omega/2\pi \tag{3.6.13}$$

rather than the angular frequency ω. From (3.6.11), therefore,

$$a(\nu) = \frac{Ne^2}{4\pi\epsilon_0 mc} \frac{\delta\nu_0}{(\nu - \nu_0)^2 + \delta\nu_0^2} \tag{3.6.14}$$

where

$$\nu_0 = \omega_0/2\pi \tag{3.6.15}$$

and

$$\delta\nu_0 = \beta/2\pi \tag{3.6.16}$$

The absorption coefficient (3.6.14) is frequently written in the form

$$a(\nu) = \frac{Ne^2}{4\epsilon_0 mc} L(\nu) \tag{3.6.17}$$

where the lineshape function $L(\nu)$ is defined by

$$L(\nu) = \frac{\delta\nu_0/\pi}{(\nu - \nu_0)^2 + \delta\nu_0^2} \tag{3.6.18}$$

This is called the *Lorentzian function*, and is plotted in Figure 3.8.

The Lorentzian function is a mathematically idealized lineshape in several respects. We have already shown that it is the near-resonance approximation to the more complicated function (3.3.25). The Lorentzian function is defined mathematically for negative frequencies, even though they have no physical significance. It is exactly normalized to unity when integrated over all frequencies, as is easily checked:

$$\int_{-\infty}^{\infty} d\nu\, L(\nu) = \frac{\delta\nu_0}{\pi} \int_{-\infty}^{\infty} \frac{d\nu}{(\nu - \nu_0)^2 + \delta\nu_0^2} = 1 \tag{3.6.19}$$

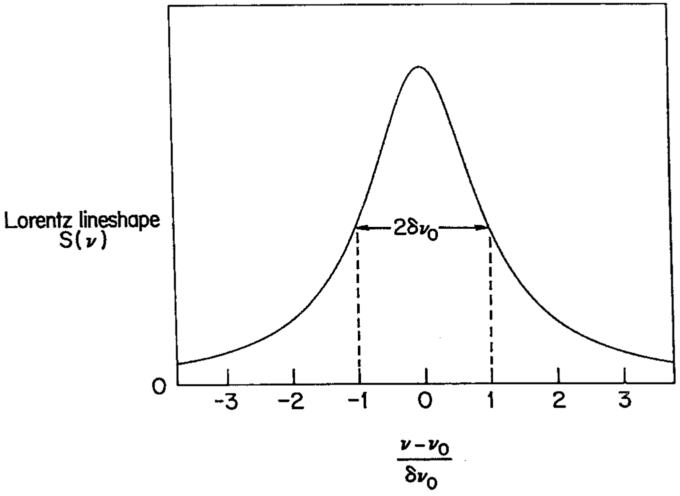

Figure 3.8 The Lorentzian lineshape function.

and the normalization is approximately the same when only the physical, positive frequencies are used. The approximation is excellent for $\delta\nu_0 \ll \nu_0$ [recall (3.3.10)]. In other words, the contribution of the unphysical negative frequencies is negligible because the linewidth is negligible compared with the resonance frequency, and in this sense $L(\nu)$ is physically as well as mathematically normalized to unity.

The maximum value of $L(\nu)$ occurs at the resonance $\nu = \nu_0$:

$$L(\nu)_{\max} = L(\nu_0) = \frac{1}{\pi\delta\nu_0} \tag{3.6.20}$$

At $\nu = \nu_0 \pm \delta\nu_0$ we have

$$L(\nu_0 \pm \delta\nu_0) = \frac{1}{2\pi\delta\nu_0} = \frac{1}{2} L(\nu)_{\max} \tag{3.6.21}$$

Because of this property, $2\delta\nu_0$ is called the width of the Lorentzian function or the *full width at half maximum* (FWHM), and $\delta\nu_0$ is called the *half width at half maximum* (HWHM). The Lorentzian function is fully specified by its width (FWHM or HWHM) and the frequency ν_0 where it peaks. The absorption coefficient is greatest at resonance, where

$$a(\nu = \nu_0) = \frac{Ne^2}{4\pi\epsilon_0 mc\delta\nu_0} \tag{3.6.22}$$

and decreases to half this resonance value when the field is "detuned" from resonance by the half width $\delta\nu_0$ of the Lorentzian function.

Our classical theory thus predicts that the absorption is strongest when the frequency of the light equals one of the natural oscillation frequencies of the bound electrons. Far out in the wings of the Lorentzian, where $|\nu - \nu_0| \gg \delta\nu_0$, there is very little absorption. A knowledge of the width $\delta\nu_0$ is therefore essential to a quantitative interpretation of absorption data. In order to determine the numerical magnitude of $\delta\nu_0$ in a given situation, we must consider in some detail the physical origin of this absorption width. This we do in Section 3.9.

We shall see later that $a(\nu)$ does not always have the Lorentzian form (3.6.17). However, it will always be possible to write the absorption coefficient as

$$a(\nu) = a_t S(\nu) \tag{3.6.23}$$

where the lineshape function $S(\nu)$, whatever its form, is normalized to unity:

$$\int_0^\infty d\nu S(\nu) = 1 \tag{3.6.24}$$

With this normalization it follows that

$$\int_0^\infty d\nu a(\nu) = a_t \int_0^\infty d\nu S(\nu) = a_t \tag{3.6.25}$$

The integrated absorption coefficient a_t is convenient because it is independent of the lineshape function $S(\nu)$, which may vary with parameters like pressure, temperature, etc. It thus provides a measure of the inherent absorbing strength of the atoms.

3.7 OSCILLATOR STRENGTH

Even more than the integrated absorption coefficient, the integrated absorption cross section, namely

$$\sigma_t = a_t/N \tag{3.7.1}$$

is a convenient measure of absorption, because it characterizes the inherent absorbing strength of a single atom. From (3.6.23) and (3.6.17) we see that

$$a_t = \frac{Ne^2}{4\epsilon_0 mc} \tag{3.7.2}$$

and therefore

$$\sigma_t = \frac{e^2}{4\epsilon_0 mc} \tag{3.7.3}$$

The numerical value of σ_t is easily computed to be approximately 2.65×10^{-2} cm^2-sec^{-1}. This is a universal value, and according to our theory is applicable to absorption by any atomic material.

Extensive experimental absorption data exist for atomic hydrogen. For example, it is known to absorb strongly at a wavelength of about 1216 Å, with an integrated absorption coefficient of about 1.1×10^{-2} cm^2-sec^{-1}. Thus our classical electron oscillator theory gives a reasonable order of magnitude, although it is far from being quantitatively accurate. However, atoms do not absorb at only one wavelength. Table 3.1 lists some wavelengths at which atomic hydrogen absorbs radiation. Our classical theory gives an integrated absorption cross section (3.7.3) which is independent of ν_0, so that the same numerical value (2.65×10^{-2} cm^2-sec^{-1}) should apply to every wavelength listed in Table 3.1. The second column of Table 3.1 lists the observed integrated cross sections of these absorption lines, while the third column gives the ratio of the observed value for each line to the result (3.7.3) of the classical theory. We see that the classical result comes close to the integrated absorption cross section only for the 1216-Å line.

Before the advent of the quantum theory, this quantitative failure of the classical theory was sidestepped by writing the integrated absorption cross section of a one-electron atom as

$$\sigma_t = \frac{e^2}{4\epsilon_0 mc} f \tag{3.7.4}$$

where the parameter f is called the "oscillator strength," and its values are given by the third column of Table 3.1. In other words, the classical theory was patched up by assigning a different "oscillator strength" to each natural oscillation fre-

TABLE 3.1 Some Integrated Cross Sections of Atomic Hydrogen

Wavelength (Å)	σ_t (actual) (cm^2-sec^{-1})	$f = \dfrac{\sigma_t \text{ (actual)}}{\sigma_t \text{ (classical theory)}}$
1216	1.10×10^{-2}	0.416
1026	2.10×10^{-3}	0.079
973	7.69×10^{-4}	0.029
950	3.71×10^{-4}	0.014
938	2.07×10^{-4}	0.0078
931	1.27×10^{-4}	0.0048

quency. In fact the integrated absorption cross section for any atom could be written in the form (3.7.4) by making the *ad hoc* replacement

$$\frac{e^2}{m} \rightarrow \frac{e^2}{m} f \tag{3.7.5}$$

wherever the quantity on the left appeared. In this way the Lorentz theory (of both absorption and the refractive index) was brought into detailed numerical agreement with experimental results. We will include f in most classical formulas hereafter. Like the natural oscillation frequencies, however, the oscillator strengths had to be taken as empirical parameters, without a theoretical basis. Quantum theory removes both of these defects of Lorentz's model.

3.8 ABSORPTION OF BROADBAND LIGHT

The rate at which the energy W_A of an atom increases due to absorption of electromagnetic energy may be obtained from (3.6.2) or (3.6.9):

$$\begin{aligned}\frac{dW_A}{dt} = \frac{-dW_{\text{em}}}{dt} &= \frac{\pi e^2 f}{2\epsilon_0 mc} \frac{\beta/\pi}{(\omega - \omega_0)^2 + \beta^2} I \\ &= \frac{\pi e^2 f}{2\epsilon_0 mc} \left(\frac{1}{2\pi} S(\nu)\right) I_\nu \end{aligned} \tag{3.8.1}$$

where we have added the subscript ν to remind us that I_ν refers to the intensity of monochromatic radiation at the frequency ν.

Equation (3.8.1) gives the rate of increase of the energy of an atom due to absorption from a monochromatic field of frequency ν. In reality, of course, the applied field will not be perfectly monochromatic. Hereafter we will indicate explicitly the dependence of field quantities on the frequency: $W_{\text{em}} \rightarrow W_{\text{em}}^{\nu}$ and $I \rightarrow I_\nu$. The change in atomic energy is due to the action of all the frequency components:

$$\begin{aligned}\left(\frac{dW_A}{dt}\right)_{\text{total}} &= \sum_\nu \left(\frac{-dW_{\text{em}}^{\nu}}{dt}\right) \\ &= \frac{e^2 f}{4\epsilon_0 mc} \sum_\nu S(\nu) I_\nu \end{aligned} \tag{3.8.2}$$

In many cases of interest the field is composed of a continuous range of frequencies, and the summation in (3.8.2) must be replaced by an integral:

$$\left(\frac{dW_A}{dt}\right)_{\text{total}} \rightarrow \frac{e^2 f}{4\epsilon_0 mc} \int_0^\infty S(\nu)\, I(\nu)\, d\nu \tag{3.8.3}$$

where $I(\nu)\, d\nu$ is the intensity of radiation in the frequency band from ν to $\nu + d\nu$.

It is convenient to define a spectral energy density $\rho(\nu)$, such that $\rho(\nu)\, d\nu$ is the electromagnetic energy per unit volume in the same frequency band (Figure 3.9). The total electromagnetic energy per unit volume is then

$$\int_0^\infty \rho(\nu)\, d\nu = \frac{1}{c} \int_0^\infty I(\nu)\, d\nu \tag{3.8.4}$$

Clearly (3.8.3) may be rewritten

$$\left(\frac{dW_A}{dt}\right)_{\text{total}} = \frac{e^2 f}{4\epsilon_0 m} \int_0^\infty S(\nu)\, \rho(\nu)\, d\nu \tag{3.8.5}$$

We can now define "broadband" light as follows. Whenever the spectral energy density $\rho(\nu)$ is a flat, almost constant function of ν compared with the atomic lineshape function $S(\nu)$, we can write

$$\begin{aligned}\int_0^\infty d\nu\, S(\nu)\, \rho(\nu) &\approx \rho(\nu_0) \int_0^\infty d\nu\, S(\nu) \\ &= \rho(\nu_0) \end{aligned} \tag{3.8.6}$$

If $\rho(\nu)$ is perfectly constant, then of course (3.8.6) is an equality. Whether $\rho(\nu)$ is flat enough in its variation to justify the approximation (3.8.6) depends on the lineshape function $S(\nu)$. The narrower the width of $S(\nu)$, the easier it is to satisfy (3.8.6). When this approximation is valid we may say that we have broadband light and broadband absorption, as opposed to the opposite extreme of narrowband (i.e., monochromatic) absorption. Both extremes are limiting cases of (3.8.5).

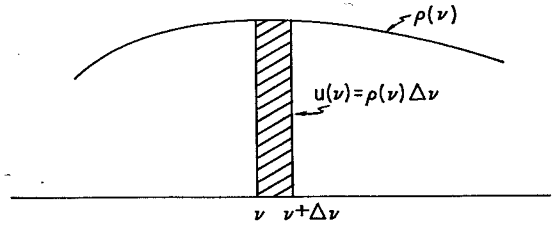

Figure 3.9 The spectral energy density $\rho(\nu)$ is defined so that $u(\nu) = \rho(\nu)\, \Delta\nu$ is the electromagnetic energy per unit volume in the narrow frequency interval from ν to $\nu + \Delta\nu$.

Therefore, the energy absorption rate for an atom exposed to broadband radiation is

$$\frac{dW_A}{dt} = \frac{e^2 f}{4\epsilon_0 m}\rho(\nu_0) \tag{3.8.7}$$

We see that for broadband absorption the rate at which the energy of the atom increases is completely independent of the form of the lineshape function $S(\nu)$, and is simply proportional to the spectral energy density of the field at the dipole's natural oscillation frequency ν_0.

In Table 3.2 we collect the most important results of this section. For simplicity we omit the background refractive index from the equations. This is always an excellent approximation for gaseous media, where n_b is close to unity. However, for solid media the index must be included. We return to this point in our discussion of laser gain in Chapter 10. We have furthermore refrained from specifying the form of the lineshape function $S(\nu)$; the question of different lineshapes is taken up in the following sections. The equations of Table 3.2 are valid for any lineshape function.

TABLE 3.2 Results of the Classical Theory of Absorption by a Medium with N Atoms per Unit Volume.

Energy Absorption Rate of an Atom

$$\frac{dW_A}{dt} = \frac{e^2 f}{4\epsilon_0 m}\int_0^\infty d\nu\, S(\nu)\,\rho(\nu) \qquad (f = \text{oscillator strength})$$

$$\approx \frac{e^2 f}{4\epsilon_0 mc} S(\nu)\, I_\nu \qquad (\text{narrowband radiation})$$

$$\approx \frac{e^2 f}{4\epsilon_0 m}\rho(\nu_0) \qquad (\text{broadband radiation})$$

Lineshape Function

$S(\nu)$ peaks at the resonance frequency $\nu = \nu_0$ and

$$\int_0^\infty d\nu\, S(\nu) = 1$$

Attenuation of Intensity for Radiation of Frequency ν

$$I_\nu(z) = I_\nu(0)\, e^{-a(\nu)z}$$

$$a(\nu) = \frac{Ne^2 f}{4\epsilon_0 mc} S(\nu) \qquad (\text{absorption coefficient})$$

$$a_t = \frac{Ne^2 f}{4\epsilon_0 mc} \qquad (\text{integrated absorption coefficient})$$

The oscillator strength f has been included by making the replacement (3.7.5), $e^2/m \rightarrow e^2 f/m$.

3.9 COLLISIONS AND "FRICTION" IN THE LORENTZ MODEL

In the preceding sections of this chapter we have shown that light is strongly absorbed when it is nearly resonant with one of the natural oscillation frequencies of the molecules of a medium, and that absorption is due to "frictional" processes that damp out dipole oscillations. We have also shown that any frictional force in the Newton equation of an electron oscillator leads to a broadened absorption line, the lineshape being Lorentzian. We did not, however, give any fundamental explanation for the existence of frictional processes. We will now approach the question of absorption and lineshape from a more fundamental viewpoint, focusing our attention on "line broadening" mechanisms in gases, in order to answer the question of the origin of the frictional coefficient β.

It is a well-known result of experiment that, for sufficiently large pressures, the width of an absorption line in a gas increases as the pressure increases. This broadening is due to collisions of the molecules and is therefore called collision broadening, or sometimes pressure broadening. Collision broadening is the most important line-broadening mechanism in gases at atmospheric pressures, and is often dominant at much lower pressures as well. We will begin our study by considering the details of collision broadening.

Our treatment of collision broadening will follow the original approach of Lorentz. We will find, for instance, that a kind of frictional force arises naturally as a result of collisions, and that the damping rate β can be interpreted as simply the collision rate.

Let us consider the effect of collisions on an atom in the electric field of a laser beam. We imagine collisions to occur in billiard-ball fashion, each collision lasting for a time that is very short compared with the time between collisions. We suppose that, immediately prior to a collision, the active electrons in an atom are oscillating along the axis defined by the field polarization, as indicated by (3.3.13). During a collision, the interaction between the two atoms causes a reorientation of the axes of oscillation. Since each atom in a gas may be bombarded by other atoms from any direction, we can assume that *on the average* all orientations of the displacements and velocities of the atomic electrons are equally probable following a collision. This is the assumption made by Lorentz. It is an assumption about the statistics of a large number of collisions, rather than about the details of a single collision.

Consider a gas of atoms at a given time t. Most atoms are not at this moment involved in a collision. Consider in particular those atoms that underwent their most recent collision at the earlier time t_1. According to our (Lorentz's) assumption, the average of the electron displacements and velocities for these atoms vanished at the time t_1, since all orientations of displacement and velocity vectors were equally probable immediately after their collision. We assume that the electrons in those atoms that had their last collision at time t_1 obey the Newton–Lorentz equation (2.3.7), which we write here in complex notation:

$$\frac{d^2\mathbf{x}}{dt^2} + \omega_0^2\mathbf{x} = \hat{\boldsymbol{\epsilon}}\,\frac{e}{m}\,E_0\,e^{-i(\omega t - kz)} \tag{3.9.1}$$

The electron displacement for a dipole satisfying this equation is obtained by combining the homogeneous and particular solutions in such a way that $\mathbf{x}(t)$ obeys the initial conditions

$$\mathbf{x}(t_1) = \left(\frac{d\mathbf{x}}{dt}\right)_{t=t_1} = 0 \tag{3.9.2}$$

Note that these are initial conditions applying to the "average" atom, since we have assumed that all displacements and velocities are equally likely after a collision. The corresponding solution to (3.9.1) will be written

$$\mathbf{x}(t;\,t_1) = \hat{\boldsymbol{\epsilon}}\,\frac{eE_0/m}{\omega_0^2-\omega^2}\left[e^{-i\omega t} - \frac{1}{2}\left(1+\frac{\omega}{\omega_0}\right)e^{-i\omega_0(t-t_1)}\,e^{-i\omega t_1} - \frac{1}{2}\left(1-\frac{\omega}{\omega_0}\right)e^{i\omega_0(t-t_1)}\,e^{-i\omega t_1}\right]e^{ikz} \tag{3.9.3}$$

It is easy to verify that (3.9.3) is the desired solution, by checking that it satisfies both (3.9.1) and the initial conditions (3.9.2). This solution will now be taken to represent the average atom. It has this average significance even if it is not applicable to any one of the atoms individually.

We wish to calculate the average electron displacement at time t for atoms in the gas, no matter when their last collision. We can obtain this by summing (3.9.3) over all possible t_1. We only need to know (at time t) the fraction $df(t,\,t_1)$ of atoms for which the last collision occurred between t_1 and $t_1 + dt_1$. We show below that this is given by

$$df(t,\,t_1) = e^{-(t-t_1)/\tau}\,\frac{dt_1}{\tau} \tag{3.9.4}$$

where τ is the mean time between collisions. The average electron displacement $\langle\,\mathbf{x}(t)\,\rangle$ for any atom at time t is therefore obtained by multiplying (3.9.3) by the fraction (3.9.4) of atoms to which it applies, and then summing (integrating) over all possible values of earlier times t_1:

$$\begin{aligned}\langle\,\mathbf{x}(t)\,\rangle &= \int_{-\infty}^{t}\mathbf{x}(t;\,t_1)\,df(t,\,t_1)\\ &= \hat{\boldsymbol{\epsilon}}\,\frac{eE_0/m}{\omega_0^2-\omega^2}\,e^{-i(\omega t-kz)}\left(\frac{1}{\tau}\right)\int_{-\infty}^{t}dt_1\,e^{-(t-t_1)/\tau}\\ &\quad\times\left[1-\frac{1}{2}\left(1+\frac{\omega}{\omega_0}\right)e^{-i(\omega_0-\omega)(t-t_1)}-\frac{1}{2}\left(1-\frac{\omega}{\omega_0}\right)e^{i(\omega_0+\omega)(t-t_1)}\right]\end{aligned} \tag{3.9.5}$$

The required integrals are

$$\int_{-\infty}^{t}dt_1\,e^{-(t-t_1)/\tau} = \tau \tag{3.9.6a}$$

$$\int_{-\infty}^{t}dt_1\,e^{-i(\omega_0-\omega)(t-t_1)}\,e^{-(t-t_1)/\tau} = \frac{-i}{\omega_0-\omega-i/\tau} \tag{3.9.6b}$$

$$\int_{-\infty}^{t}dt_1\,e^{i(\omega_0+\omega)(t-t_1)}\,e^{-(t-t_1)/\tau} = \frac{i}{\omega_0+\omega+i/\tau} \tag{3.9.6c}$$

The average electron displacement is therefore given by

$$\begin{aligned}\langle\,\mathbf{x}(t)\,\rangle &= \hat{\boldsymbol{\epsilon}}\,\frac{eE_0/m}{\omega_0^2-\omega^2}\,e^{-i(\omega t-kz)}\\ &\quad\times\left[1+\frac{i}{2\tau}\,\frac{1+\omega/\omega_0}{\omega_0-\omega-i/\tau}-\frac{i}{2\tau}\,\frac{1-\omega/\omega_0}{\omega_0+\omega+i/\tau}\right]\\ &= \frac{\hat{\boldsymbol{\epsilon}}(eE_0/m)\,e^{-i(\omega t-kz)}}{\omega_0^2-\omega^2-2i\omega/\tau+1/\tau^2}\end{aligned} \tag{3.9.7}$$

and the corresponding polarizability is

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2-\omega^2-2i\omega/\tau+1/\tau^2}. \tag{3.9.8}$$

Note that, except for the term $1/\tau^2$, this is the same as (3.3.17) if we identify the frictional coefficient β with the collision rate $1/\tau$.

The main conclusion to be drawn from our collision analysis is obvious. Given the strong inequality

$$\omega\tau \gg 1 \tag{3.9.9}$$

which implies that the mean time between collisions is much longer than an optical period ($\sim 10^{-15}$ sec), and which is an excellent approximation in practice, the last term in the denominator of (3.9.8) can be dropped. Then the effect of collisions is exactly the same as the effect of a frictional damping force if we let

$$\beta = \frac{1}{\tau} = \text{collision rate} \tag{3.9.10}$$

However, we must not lose sight of the statistical nature of our treatment of collisions. We should really say that a frictional term in the Newton equation is justified by the effects of collisions *on the average*. Thus we can give up the artificial notion of friction at the atomic level, but still use all of the results derived from it, if we reinterpret $\mathbf{x}(t)$, U, V, and W_A in Sections 3.5 and 3.6 as *average* values in the sense of (3.9.5). We are thus led to regard the results of Table 3.2 with the Lorentzian lineshape function (3.6.18) as the consequences of collision broadening. The width (HWHM) of this collision-broadened lineshape function is

$$\delta\nu_0 = \frac{\beta}{2\pi} = \frac{1}{2\pi\tau} \tag{3.9.11}$$

The damping term we introduced empirically earlier in (3.2.3) may now be interpreted as the damping of the average electron displacement, i.e.,

$$\frac{d^2}{dt^2}\langle \mathbf{x} \rangle + 2\beta \frac{d}{dt}\langle \mathbf{x} \rangle + \omega_0^2 \langle \mathbf{x} \rangle = \hat{\varepsilon}\frac{e}{m}E_0\, e^{-i(\omega t - kz)} \tag{3.9.12}$$

Collision broadening is often described equivalently in terms of a "dephasing" of the electron oscillators, as follows. Immediately after a collision the phase of the electron's oscillation has no correlation with the precollision phase. Collisions have the effect of "interrupting" the phase of oscillation, leading to an overall decay of the average electron displacement from equilibrium (Figure 3.10). The damping rate in (3.9.12) is sometimes called a "dephasing" rate, in order to distinguish it from an "energy decay" rate. The latter would appear as a frictional term in the equation of motion of each electron oscillator as well as in the average equation. In the absence of any inelastic collisions to decrease the energy of the electron oscillators, each oscillator would satisfy the Newton equation (2.3.7) with no damping term. Due to elastic collisions, i.e., collisions which only interrupt the phase of oscillation but do not produce any change in energy, the *average* electron displacement follows equation (3.9.12), which includes damping.

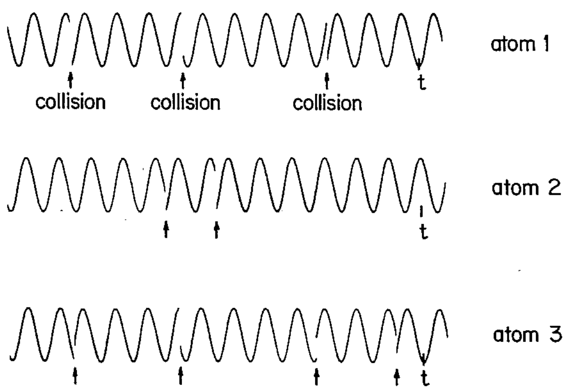

Figure 3.10 Electron oscillations in three different atoms in a gas. Collisions completely interrupt the phase of the oscillation. The average electron displacement associated with all the atoms in the gas therefore decays to zero at a rate given by the inverse of the mean collision time.

- To complete our derivation of (3.9.7), we must prove our assertion (3.9.4). The mean time between collisions, τ, is obviously an average; a given atom certainly does not have collisions in evenly spaced intervals of time τ. We can only say that the *probability* of any given atom having a collision in a small time interval Δt is given by Δt times the mean number of collisions per unit time, $1/\tau$. If at time T there are $\eta(T)$ atoms which have not yet had a collision since the time $T = 0$, then the number of "collisionless" atoms at time $T + \Delta T$ is

$$\eta(T + \Delta T) = \eta(T) - \eta(T)\frac{\Delta T}{\tau} \tag{3.9.13}$$

In words, $\eta(T)$ decreases by the amount $\eta(T)\,\Delta T/\tau$, which is the number of atoms collisionless since the time $T = 0$, times the probability that any one such atom will have a collision in the time interval ΔT. Thus

$$\frac{\eta(T + \Delta T) - \eta(T)}{\Delta T} = -\frac{\eta(T)}{\tau} \tag{3.9.14}$$

The limit $\Delta T \to 0$ gives the simple differential equation

$$\frac{d\eta(T)}{dT} = -\frac{1}{\tau}\eta(T) \tag{3.9.15}$$

with the solution

$$\eta(T) = \eta_0 e^{-T/\tau} \tag{3.9.16}$$

Since η_0 is the total number of atoms in the gas, the quantity

$$P(T) = e^{-T/\tau} \tag{3.9.17}$$

is the probability that a given atom has had no collision for a time T. At $T = 0$, when we begin "looking," this probability is unity. For $T \gg \tau$ it is very small, because in all likelihood the atom will have a collision before many collision times τ have elapsed.

The probability that a given atom will have no collision for a time T, and then have a collision within a time interval dT, is just the product of the probabilities for these two "events", i.e., $P(T)\,dT/\tau$. This is just the fraction $df(T)$, of the total number of atoms, that can be expected to have their *next* collision within the time interval from T to $T + dT$, after we begin "looking" at $T = 0$. If we imagine a movie showing the movements and collisions of the atoms, we can run our film backwards in time and the collisions will exhibit the same statistical behavior. And we will observe the same statistical behavior regardless of where we begin looking.

The atoms at time t that had their last collisions in the interval from t_1 to $t_1 + dt_1$ will be just those having their *next* collision in the same interval when we look at the gas backwards in time beginning at time t. Thus the fraction $df(t, t_1)$ of atoms at time t that had their last collision in an interval dt_1 of $t_1 < t$ will be the same as the fraction of atoms at time t which will have their next collision in an interval dt_1 of t_1 when the film is run backwards. This is just the probability $P(T)\,dt_1/\tau$ found above, with $T = t - t_1$. Thus

$$df(t, t_1) = P(t - t_1)\frac{dt_1}{\tau} = e^{-(t-t_1)/\tau}\frac{dt_1}{\tau} \qquad (3.9.18)$$

which is the same as (3.9.4). •

3.10 COLLISION CROSS SECTIONS

We have shown in the preceding section that collisions, on average, can produce the same effect as frictional damping on an electron oscillator. The damping rate β can be identified with the collision rate $1/\tau$. Therefore the magnitude of $1/\tau$ is of direct significance for realistic estimates of line broadening.

The collision rate $1/\tau$ may be expressed in terms of the number density N of atoms, the collision cross section $\bar{\sigma}$ between atoms, and the average relative velocity $\bar{v}$ of the atoms. Imagine some particular atom to be at rest and bombarded by a stream of identical atoms of velocity $\bar{v}$. If the number of atoms per unit volume in the stream is N, then the number of collisions per unit time undergone by the atom at rest is $N\sigma\bar{v}$, where the area σ is the *collision cross section* between the atom at rest and an atom in the stream. The number of collisions per second is the same as if all the stream atoms within a cross-sectional area σ collide with the stationary atom. The idea here is exactly the same one used to define scattering and absorption cross sections for incident light.

According to the kinetic theory of gases, an atom of mass m_x has an rms velocity

$$v_{\text{rms}} = \left(\frac{8kT}{\pi m_x}\right)^{1/2} \qquad (3.10.1)$$

in a gas in thermal equilibrium at temperature T, where k is Boltzmann's constant.

To obtain the average relative velocity $\bar{v}_{\text{rel}}$ of colliding atoms of masses m_x and m_y in the gas, we replace m_x in (3.10.1) by the reduced mass

$$\mu_{x,y} = \frac{m_x m_y}{m_x + m_y} = \left(\frac{1}{m_x} + \frac{1}{m_y}\right)^{-1} \qquad (3.10.2)$$

Thus

$$\bar{v}_{\text{rel}} = \left[\frac{8kT}{\pi}\left(\frac{1}{m_x} + \frac{1}{m_y}\right)\right]^{1/2} \qquad (3.10.3)$$

It is convenient to express this in terms of the atomic (or molecular) weights M_x and M_y:

$$\bar{v}_{\text{rel}} = \left[\frac{8RT}{\pi}\left(\frac{1}{M_x} + \frac{1}{M_y}\right)\right]^{1/2} \qquad (3.10.4)$$

where R, the universal gas constant, is Boltzmann's constant times Avogadro's number. The collision rate for molecules of type x is therefore

$$\begin{aligned}\frac{1}{\tau} &= \sum_Y N(Y)\sigma(X, Y)\,\bar{v}_{\text{rel}}(X, Y)\\ &= \sum_Y N(Y)\sigma(X, Y)\left[\frac{8RT}{\pi}\left(\frac{1}{M_x} + \frac{1}{M_y}\right)\right]^{1/2}\end{aligned} \qquad (3.10.5)$$

where the sum is over all species y, including x.

The important "unknowns" in the expression (3.10.5) are the collision cross sections $\sigma(X, Y)$. It often happens that these are not known very accurately. They are difficult to derive theoretically, and experimental determinations are not always unambiguous. The simplest approximation to the cross section is the "hard-sphere" approximation. We write

$$\bar{\sigma}(X, Y) = \frac{\pi}{4}(d_x + d_y)^2 \qquad (3.10.6)$$

where d_x and d_y are the "hard-sphere" molecular diameters, estimates of which are sometimes tabulated. $\bar{\sigma}(X, Y)$ is just the area of a circle of diameter $d_x + d_y$, just what we would expect if the molecules acted like spheres of diameters d_x and d_y. For CO_2, for example, the hard-sphere diameter is about 4.00 Å. From (3.10.6), therefore, the hard-sphere cross section for two CO_2 molecules is $\bar{\sigma}(CO_2, CO_2) = 5.03 \times 10^{-15}\ \text{cm}^2$. For a gas of pure CO_2 at $T = 300$ K we find the average relative velocity of two colliding CO_2 molecules to be $\bar{v}_{\text{rel}} = 5.37 \times 10^4$ cm/sec. The collision rate (3.10.5) in the "hard-sphere" approximation is therefore

$$\frac{1}{\tau} = N(5.03 \times 10^{-15}\ \text{cm}^2)(5.37 \times 10^4\ \text{cm/sec})$$

$$= 2.70 \times 10^{-10} N/\text{sec} \tag{3.10.7}$$

where N is the number of CO_2 molecules in a cubic centimeter. For an ideal gas we calculate (Problem 3.5)

$$N = 9.65 \times 10^{18} \frac{P(\text{Torr})}{T} \tag{3.10.8}$$

where $P(\text{Torr})$ is the pressure in Torr (1 atmosphere = 760 Torr) and T is the temperature (K). From (3.10.7), finally, the collision rate for a gas of CO_2 at 300 K is

$$\frac{1}{\tau} = 8.69 \times 10^6 P(\text{Torr})\ \text{sec}^{-1} \tag{3.10.9}$$

Thus at a pressure of 1 atmosphere (760 Torr) we calculate the collision rate

$$\frac{1}{\tau} = 6.60 \times 10^9\ \text{sec}^{-1} \tag{3.10.10}$$

and from (3.9.11) the collision-broadened linewidth

$$\delta\nu_0 = 1.05 \times 10^9\ \text{Hz} \tag{3.10.11}$$

The actual collision-broadened linewidths can be larger, by as much as an order of magnitude or more, than those calculated in the hard-sphere approximation. The value calculated above, however, is reasonable, and it allows us to point out some general features of the collision-broadened linewidths. First we note that the collision rate (3.10.10) is very much smaller than an optical frequency, as assumed in (3.9.9). The linewidth $\delta\nu_0$ is thus also orders of magnitude less than an optical frequency. This explains why we can speak of absorption "lines" in a gas, even though the absorption occurs over a band of frequencies: the band has a width ($\sim 2\delta\nu_0$) that is very small compared with the resonance frequency ν_0.

From (3.10.9) we note that the linewidth is linearly proportional to the pressure. For this reason, experimental results for collision-broadened linewidths are often reported in units such as MHz-Torr^{-1}. The linewidth calculated above, for instance, may be expressed as 1.38 MHz-Torr^{-1} at 300 K.

Our treatment of collision broadening only highlights some general features of a complex subject. In actual calculations we prefer always to use measured values of the collision-broadened linewidths. We note parenthetically that, for the 10.6-μm CO_2 laser line, the linewidth (1.38 MHz-Torr^{-1}) computed above is about three times smaller than the experimentally determined value. It is possible to calculate these widths more accurately, but this will not concern us.

3.11 DOPPLER BROADENING

The Doppler effect was demonstrated for sound waves in 1845 by C. H. D. Buys Ballot, who employed trumpeters performing in a moving train to demonstrate it. The mathematician C. J. Doppler had predicted the effect in 1842. His prediction applied also to light, although Maxwell's electromagnetic theory of light waves was still nearly a quarter of a century away.

Let us consider again a gaseous medium, this time only very weakly influenced by collisions (i.e., β is very small). Every electron oscillator will thus undergo practically undamped oscillation at the field frequency. Nevertheless we will show that, because of the Doppler effect, an absorption line is broadened and its width can be much larger than β. We will find that the lineshape associated with the Doppler effect is not the Lorentzian function (3.6.18), but rather the Gaussian function given in (3.11.6) below.

To an atom moving with velocity $v \ll c$ away from a source of radiation of frequency ν, the frequency of the radiation appears to be shifted:

$$\nu' = \nu\left(1 - \frac{v}{c}\right) \tag{3.11.1}$$

This is the Doppler effect. It implies that a source of radiation (e.g., a laser) exactly resonant in frequency with an absorption line of a stationary atom will not be in resonance with the same absorption line in a moving atom, and the frequency offset is $\delta\nu = (v/c)\nu$. Similarly, a nonresonant absorption line of an atom may be brought *into* resonance with the field as a result of atomic motion. Since the atoms in a gas exhibit a wide variety of velocities, a broad range of different effective resonance frequencies will be associated with a given absorption line. In other words, the absorption line is broadened because of the Doppler effect. The absorption line is thus said to be Doppler-broadened.

For a gas in thermal equilibrium at the temperature T, the fraction $df(v)$ of atoms having velocities between v and $v + dv$ along any one axis is given by the (one-dimensional) Maxwell–Boltzmann distribution,

$$df(v) = \left(\frac{m_x}{2\pi kT}\right)^{1/2} e^{-m_x v^2/2kT} dv \tag{3.11.2}$$

Here again k is the Boltzmann constant and m_x is the mass of an atom or molecule. Because we have assumed that collisions are almost negligible, an atom with resonance frequency ν_0 and velocity v moving away from the source of radiation will only absorb radiation very near to (within $\Delta\nu = \beta/2\pi$ of) the frequency

$$\nu = \nu_0\left(1 + \frac{v}{c}\right) \tag{3.11.3}$$

The fraction of atoms absorbing within the frequency interval from ν to $\nu + d\nu$ is thus equal to the fraction of atoms with velocity in the interval from v to $v + dv$. From (3.11.3) we have

$$v = \frac{c}{\nu_0}(\nu - \nu_0) \tag{3.11.4}$$

and $dv = (c/\nu_0)\, d\nu$. Using (3.11.2) we can determine that this fraction is

$$df_\nu(v) = \left(\frac{m_x}{2\pi kT}\right)^{1/2} e^{-m_x c^2(\nu-\nu_0)^2/2kT\nu_0^2}\left(\frac{c}{\nu_0}\, d\nu\right) \tag{3.11.5}$$

Since the absorption rate at frequency ν must be proportional to $df_\nu(v)$, we may write the Doppler lineshape function as

$$S(\nu) = \left(\frac{m_x c^2}{2\pi kT\nu_0^2}\right)^{1/2} e^{-m_x c^2(\nu-\nu_0)^2/2kT\nu_0^2} \tag{3.11.6}$$

Since (3.11.2) was normalized to unity when integrated over velocity, (3.11.6) is normalized to unity with respect to the frequency offset (or "detuning") $\nu - \nu_0$, as required by the definition of lineshape function.

By direct computation using (3.11.6) we find

$$\int_0^\infty d\nu\, S(\nu) = S(\nu_0)\int_0^\infty d\nu\, e^{-m_x c^2(\nu-\nu_0)^2/2kT\nu_0^2}$$

$$= S(\nu_0)\int_{-\nu_0}^\infty d\mu\, e^{-m_x c^2\mu^2/2kT\nu_0^2}$$

$$\approx S(\nu_0)\int_{-\infty}^\infty d\mu\, e^{-m_x c^2\mu^2/2kT\nu_0^2}$$

$$= S(\nu_0)\left[\frac{\nu_0}{c}\left(\frac{2\pi kT}{m_x}\right)^{1/2}\right]$$

$$= 1 \tag{3.11.7}$$

We have used the excellent approximation $kT \ll m_x c^2$ to replace the lower limit of the integral by $-\infty$. Thus we may write

$$S(\nu) = \frac{c}{\nu_0}\left(\frac{m_x}{2\pi kT}\right)^{1/2} e^{-m_x c^2(\nu-\nu_0)^2/2kT\nu_0^2} \tag{3.11.8}$$

It is convenient to define

$$\delta\nu_D = 2\frac{\nu_0}{c}\left(\frac{2kT}{m_x}\ln 2\right)^{1/2} \tag{3.11.9}$$

in terms of which

$$S(\nu) = \frac{1}{\delta\nu_D}\left(\frac{4\ln 2}{\pi}\right)^{1/2} e^{-4(\nu-\nu_0)^2\ln 2/\delta\nu_D^2}, \tag{3.11.10}$$

and we recognize that $\delta\nu_D$ is the width (FWHM) of the Doppler absorption curve, since

$$S(\nu_0 \pm \tfrac{1}{2}\delta\nu_D) = S(\nu_0)\, e^{-\ln 2} = \tfrac{1}{2}S(\nu_0) \tag{3.11.11}$$

$\delta\nu_D$ is commonly called the *Doppler width* (Figure 3.11). The Doppler width is also often defined in terms of the $1/e$ point of the curve, rather than the half-maximum point. Sometimes it is defined as the half width at half maximum (HWHM) rather than the FWHM. Thus one finds formulas in the literature differing by factors of 2, ln 2, etc. It is important to keep these possible differences in mind when comparing calculations.

The peak of the Doppler curve at $\nu = \nu_0$ has the value

$$S(\nu_0) = \frac{1}{\delta\nu_D}\left(\frac{4\ln 2}{\pi}\right)^{1/2} \tag{3.11.12}$$

where $S(\nu_0)$ is evidently the peak value of $S(\nu)$, for which $\nu = \nu_0$. $S(\nu_0)$ is determined from the normalization condition (3.11.7).

TABLE 3.3 Collision, Doppler, and Voigt Lineshapes

Collision-Broadening Lineshape

$$S(\nu) = \frac{(1/\pi)\,\delta\nu_0}{(\nu - \nu_0)^2 + \delta\nu_0^2}$$

$$\delta\nu_0 = \frac{\text{collision rate}}{2\pi}$$

Doppler-Broadening Lineshape

$$S(\nu) = \frac{0.939}{\delta\nu_D}\, e^{-2.77(\nu-\nu_0)^2/\delta\nu_D^2}$$

$$\delta\nu_D = 2.15 \times 10^6 \left[\frac{1}{\lambda_0}\left(\frac{T}{M}\right)^{1/2}\right] \text{MHz}$$

T = gas temperature (K)

M = molecular weight of absorber (g)

λ_0 = wavelength (Å) of absorption line

Voigt Lineshape

$$S(\nu) = \frac{0.939}{\delta\nu_D}\,\mathrm{Re}\, w(x + ib)$$

$$x = 1.67\,\frac{\nu_0 - \nu}{\delta\nu_D}$$

$$b = 1.67\,\frac{\delta\nu_0}{\delta\nu_D}$$

w = error function of complex argument

3.13 EXAMPLE: ABSORPTION BY SODIUM VAPOR

Let us consider an example of the use of Tables 3.2 and 3.3. Consider the 5890-Å absorption line of sodium vapor at 300 K. The Doppler width is

$$\delta\nu_D \approx 2.15 \times 10^{12} \left[\frac{1}{5890}\left(\frac{300}{23}\right)^{1/2}\right] \text{Hz}$$
$$\approx 1300 \text{ MHz} \tag{3.13.1}$$

since the atomic weight of sodium is $M_{\text{Na}} = 23$ g. From tabulated data we can

estimate the collision broadening linewidth of the 5890-Å line in pure sodium vapor at 300 K to be

$$\delta\nu_0 \approx 1700P(\text{Torr}) \text{ MHz} \tag{3.13.2}$$

The ratio b of Table 3.3 is therefore

$$b \cong 2.2P(\text{Torr}) \tag{3.13.3}$$

If P(Torr) is less than, say, about 0.1 Torr, we are in the "Doppler regime." In this case the absorption coefficient for narrowband light exactly resonant with the 5890-Å absorption line is found from Table 3.2 at the end of Section 3.8 to be

$$a(\nu_0) = \frac{e^2 f}{4\epsilon_0 mc} N\, S(\nu_0)$$
$$= \frac{e^2 f}{4\epsilon_0 mc} N \frac{1}{\delta\nu_D}\left(\frac{4 \ln 2}{\pi}\right)^{1/2} \tag{3.13.4}$$

For the sodium D lines the oscillator strength—the factor f—is of order unity. In fact the 5890- and 5896-Å lines have oscillator strengths of 0.355 and 0.627, respectively. From (3.13.4), (3.13.1), and (3.10.8), therefore, we obtain

$$a(\nu_0) \approx 2.2 \times 10^5 P(\text{Torr}) \text{ cm}^{-1} \tag{3.13.5}$$

which is valid provided the pressure is small enough that Doppler broadening prevails. For narrowband light of frequency ν not necessarily equal to ν_0 we have

$$a(\nu) \approx 2.2 \times 10^5 P(\text{Torr})\, e^{-4(\nu-\nu_0)^2 \ln 2/\delta\nu_D^2} \text{ cm}^{-1} \tag{3.13.6}$$

In Figure 3.14 we have plotted the transmission coefficient

$$\frac{I_\nu(z)}{I_\nu(0)} = e^{-a(\nu)z} \tag{3.13.7}$$

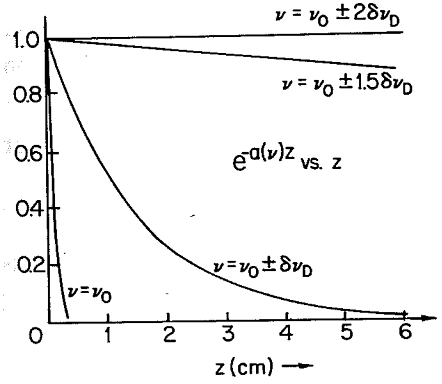

Figure 3.14 Transmission coefficient for 5890-Å radiation in sodium vapor at T = 300 K, $P = 5 \times 10^{-5}$ Torr. In this case the absorption line is Doppler-broadened, with $\delta\nu_0 \approx 1300$ MHz. The four curves illustrate the high selectivity of the absorption process.

for nearly monochromatic light of slightly different frequencies. It is evident that the transmission coefficient has an extremely strong dependence on the detuning $\nu - \nu_0$. The detuning of the field frequency ν by just a very small fraction of ν from the resonance frequency ν_0 results in a very sharp increase in the transmission coefficient. Similar results apply at higher pressures, where collision broadening becomes important.

PROBLEMS

3.1 Assume the "spring constants" k for the binding of electrons in atoms are approximately the same as those for the binding of atoms in molecules. If $\nu \approx 5 \times 10^{14}$ Hz is a typical electronic oscillation frequency, estimate the range of frequencies typical of atomic vibrations in molecules, given typical electron–atom mass differences. Does your estimate indicate that molecular vibrations lie in the infrared region of the spectrum?

3.2 The atomic weight of lithium is 6.939 g, and the density of lithium is 0.534 g/cm^3. Assuming each lithium atom contributes one electron to the "free-electron gas," calculate the plasma frequency ν_p. For what wavelengths would you expect lithium to be transparent? (Note: The transparency of the alkali metals in the ultraviolet was discovered by R. W. Wood in 1933.)

3.3 The addition of an ohmic current density to Maxwell's equations leads to the wave equation (3.5.6). Show this by adding $\mathbf{J} = \sigma\mathbf{E}$ to the right side of (2.1.4) and then retracing the derivation of the wave equation (2.1.13).

3.4* Derive the equation for classical "laser amplification" by substituting (3.5.2) into (3.5.6), allowing E_n to be time-dependent: $E_n = E_n(t)$. It is sufficient to assume that $E_n(t)$ is slowly varying so that terms proportional to d^2E_n/dt^2 can be discarded.

(a) Obtain the equation for dE_n/dt.

(b) Use the approximations and abbreviations given in Eqs. (3.5.7)–(3.5.10) to show that $d|E_n|^2/dt = 0$ if $g = \sigma/\epsilon_0 c$.

(c) Sketch on one graph the behavior of $E_n(t)$ vs. t obtained from the solution of the equation for $d|E_n|^2/dt$ under the (unrealistic) assumption that $g = 2\sigma/\epsilon_0 c$, and the (more realistic) assumption that $g = -2\sigma/\epsilon_0 c$.

3.5 Show that the number of atoms (or molecules) per cm^3 of an ideal gas at pressure P and temperature T is given by (3.10.8).

3.6 **a.** Verify Eq. (3.12.3).

b. Using Eqs. (3.4.9) and (3.4.11), show that in the absence of background atoms

$$n_R(\nu) - 1 = \frac{\lambda_0}{4\pi}\frac{\nu_0 - \nu}{\delta\nu_0}\,a(\nu), \qquad \lambda_0 = \frac{c}{\nu_0}$$

This equation relates the refractive index near a collision-broadened absorption line to the absorption coefficient.

3.7* Although the relation derived in Problem 3.6 applies to the case of collision broadening, a similar relation holds more generally. Show in the case of a Voigt profile that

$$n_R(\nu) - 1 = \frac{\lambda_0}{4\pi}\frac{\operatorname{Im} w(x + ib)}{\operatorname{Re} w(x + ib)}\,a(\nu)$$

where w, x, and b are defined in Section 3.12.

[Note: The relation between the refractive index and the absorption coefficient (or, equivalently, between the real and imaginary parts of the complex refractive index) is a special case of the so-called *Kramers–Kronig relations*. Such relations may be derived on very general grounds based on causality.]

3.8 Estimate the absorption coefficient for 5890-Å radiation in sodium vapor containing 2.7×10^{12} atoms/cm^3 at 200°C. [See J. E. Bjorkholm and A. Ashkin, *Phys. Rev. Lett.* **32,** 129 (1974)].

3.9 The CO_2 molecule has strong absorption lines in the neighborhood of $\lambda = 10\ \mu\text{m}$. Assuming that the cross sections of CO_2 molecules with N_2 and O_2 molecules are $\sigma(CO_2, N_2) = 120\ \text{Å}^2$ and $\sigma(CO_2, O_2) = 95\ \text{Å}^2$, estimate the collision-broadened linewidth for CO_2 in the earth's atmosphere. (Note: since the concentration of CO_2 is very small compared with N_2 and O_2 in air, you may assume that only N_2–CO_2 and O_2–CO_2 collisions contribute to the linewidth.) Compare this with the Doppler width.

3.10 Consider the absorption coefficient $a(\nu_0)$ of a pure gas precisely at resonance. Show that $a(\nu_0)$ is proportional to the number density of atoms when the absorption line is Doppler-broadened, but is independent of the number density when the pressure is sufficiently large that collision broadening is dominant.